WORKSHOP FOR DEVELOPMENT OF SELF-LEARNING MATERIALS ON SPECIFIC TOPICS IN MATHEMATICS AT HIGHER SECONDARY LEVEL

Report of the Programme held at

REGIONAL INSTITUTE OF EDUCATION, BHUBANESWAR

December 16–23, 1991

Dr. K. K. Chakravarti

Reader in Mathematics

PROGRAMME DIRECTOR

Department of Science and Mathematics

REGIONAL INSTITUTE OF EDUCATION

(National Council of Educational Research and Training)

BHUBANESWAR–751007

FOREWORD

Of late, the importance of developing self-learning materials on various topics in the area of Mathematics at +2 level has been realised by the teachers and the teacher-educators in order to allow the learners to learn at their own speed, own style and own time. Keeping this objective in view, three workshops were organised in Regional Institute of Education, Bhubaneswar towards development of self-learning materials on specific themes in Mathematics at +2 level during the period from 16.12.91 to 23.12.91, 23.3.92 to 28.3.92 and 21.12.92 to 28.12.92. The self-learning materials were developed on the topics like ;

i) Permutations and Combinations, Transformation Geometry, Axiomatisation of Mathematics, Rolle's Theorem, Mean Value Theorems and their Geometrical Interpretations, Applicable Mathematics I, II and III in one package;

ii) resolution of the general equation of second-degree in x and y to the appropriate conics with all its determinants in another package; and

iii) Graphs of Equalities and Inequalities in one and two Variables, graphs of Functions, First Derivative Test for Relative Maximum and Minimum, Horizontal and Vertical Asymptotes, their concavity and points of inflections including graphs of Trignometrical Functions, Exponential Functions and their Inverse Functions.

I appreciate the efforts of Dr.K.K.Chakravarti, Reader in Mathematics and the Programme Coordinator for finalising the materials for the benefit of the students of Mathematics at +2 level.

(Prof.D.K.Bhattacharjee)
Principal

THE WORK

Workshop for Development of Self-learning Materials on specific Topics in Mathematics at the +2 level was conceived by the Department of Science, Regional College of Education, Bhubaneswar to alleviate the difficulties faced by the

a) teachers in teaching some topics in the classrooms; and

b) students in understanding and then learning some topics

A study of sort, though very informally, was undertaken to identify the difficult areas from the points of view of

c) classroom teaching and work; and

d) students' understanding and learning

The study, in a nutshell, was the discussions held with the concerned groups, namely the

i) teachers of mathematics at the +2 level

ii) the students having mathematics at the +2 level in Sciences/Humanities/Commerce; and

iii) mathematicians and educationists who have had teaching experience from K.V.S. system, +2 schools, Colleges and Boards of Higher Secondary Education of some of the states of the Eastern Region;

Many topics from the +2 syllabus which are difficult naturally came up for discussions from which a tentative and suggestive list were drawn up. It was decided that

these topics will be presented in a written form which can facilitate both teaching and learning and make them better understandable to students. Most appropriately teachers were required to be the authors of these topics to provide a write up for themselves as they will need them for classroom work for the benefit of students.

The authorities of the concerned States could not provide the same set of teachers to participate in each of the three phases of the workshops. It became therefore difficult to develop materials in a sequential manner and to try them out which were the objectives to start with. Attempt was therefore directed to the development of different topics from the suggestive list which were to the liking of the participating groups in each of the workshops.

Participants of the first workshop held from 16.12.91 to 23.12.91 selected the following topics for development:

i) that part of the theory of permutations and combinations (which finds no place in books of starting algebra) which deals with and emphasises on the relation of the standard results to the generating functions. The generating functions, if understood and frame d correctly illuminates the purpose of the theory and enlarges the scope of application of it to solve more interesting problems in permutations and combinations.

ii) Geometrical and other interpretations of the well known results of Rolle's Theorem and Mean Value theorems were developed including the approximation of the value of the function f at a point x, i.e. f(x) in terms of $f(a)$, $f'(a)$, $f''(a)$, ... etc. where x is a point in the neighbourhood of a

iii) The demonstration of the applications of the simple differential equations first in developing a model and then finding its solutions.

Three participants joined the Second Phase of the programme. They also attended the first phase of the workshop. Though they were very busy with the preparation of materials of the first workshop they agreed to adopt the following suggested topic for development.

Analysis of a second degree equation in two variables x and y, the conic it represents with its centre, vertices, foci directrices etc. and its graph in the plane.

The development needed the study of

i) conics in standard forms as are given in usual text books; and

ii) translations and rotations of axes and how they can change the general equation of second degree into standard forms with respect to different sets of axes generated in a given framework of axes of a plane.

The work had been done as dealing with a second degree equation in x and y was not very easy and not dealt in ordinary text books. The equation was to be brought to the standard form with respect to some frame of axes in a plane to determine the conic it represented alongwith its determinants. The graph then could be drawn very easily in the plane with the help of its characteristics, namely, the generated frame of reference of the standard form, centre, vertices etc. The analysis was completed by converting them back into original frame of axes of x and y by utilising the equations of translations and rotations.

A new group of six teachers from the different states joined the third phase of the workshop held during December 21 - 28, 1992.

They preferred to select "Drawing of Graphs of Functions" from the suggestive list. They worked to develop

(i) graphs of equalities/inequalities of polynomial in one and two variables

(ii) graphs of standard functions

(iii) graphs of functions which could be obtained from the standard functions by translations or rotations or both. Translations and rotations were used as handy operations.

(iv) graphs of inverse functions

(v) graphs of Logarithmic functions

Indeed it was a very time consuming hard and good work. Any suggestion for the improvement of the work will be accepted with appreciation.

K. Chakravarti

ACKNOWLEDGEMENTS

The three-phase Workshop for Development of Self-Learning Materials on Specific Topics in Mathematics at the +2 Level was conducted by the Department of Science duri·g the period December December 1991 to December,1992.

The mathematics teachers of different states working at the +2 level participated in these programmes. They discussed the difficult areas in Mathematics at the Higher Secondary level which were suggested by them and others in the profession from the point of their effective teaching and of better learning by students. They have prepared the texts on some of the suggested difficult topics to facilitate better teaching in class rooms by professionals of their kind and to improve the understanding and learning by students. I appreciate their sincere endeavour in the preparation of such texts and thank them all for the same.

I take this opportunity to thank the Principal and the Head of the Department of Science for entrusting me with the work. It is also my pleasure to thank the staff of the Extension Department, Science Department, C & W Section and Accounts Section for their timely help in concerned matters.

Lastly I look forward to the effect that these work would ultimately bring to the teaching of mathematics at the +2 level.

K.K.Chakravarti

(Dr.K.K.Chakravarti)
Programme Director

APPROACH PAPER

We are very glad that the State Government has sponsored your name for participating in the "Workshop for Development of self-learning materials on special topics in mathematics at the +2 level (Higher Secondary).

This is a three-phase workshop. In the first phase we will discuss about the present syllabus of mathematics at the +2 level of your state for identification of difficult topics from the point of view of teaching or learning or both. This may take a day or a day and a half atmost. We will then get down to actual writing the topics we all consider difficult. This is for a good representation of the topics so that it facilitates better grasp and understanding of the topics by students and other users.

A methodical step by step development of each topic is therefore envisaged. The writing will be more or less problem solving oriented.

In the second phase we need your again for another spell of eight days. Then we will have to develop some more new topics in addition to revising and editing of the materials written in the first phase. Also sufficient number of problems of various types (from easier to harder) have to find place in our developed instructional materials.

contd...

In the third phase of this programme we hope to try out the materials so developed by using them in actual classroom teaching and ask for critical evaluation of the lessons from teachers.

This is in a nutshell the aim and objective of the programme I want to conduct alongwith you. You have now become an important member for developing and executing the programme in all its three phases.

I welcome you. By this time you must have realised that some preparation is needed by us all for active participation in the programme.

In the ensuing three-phase programme we will work on Class-XI syllabus only. Therefore, before starting for Bhubaneswar, have a discussion with your colleagues who teach mathematics at +2 level and specially those who take classes in XI standard. Their valued opinion in the matter will be of immense help to us. While coming please bring the syllabii of Class XI and XII of your state Council alongwith some well-known and standard text books which are generally followed in the class. I may remind you that these are essential for the purpose of our work.

Hope to meet you soon. With thanks.

K.K.Chakravarti

(Dr.K.K.Chakravarti)
Programme Director

C O N T E N T S

*****

WORKSHOP FOR DEVELOPMENT OF SELF-LEARNING MATERIALS ON SPECIFIC TOPICS ON MATHEMATIES AT + 2 LEVEL HELD FROM 16.12.91 TO 23.12.91

LIST OF PARTICIPANTS

1. Sri Babu Lal Jha
   Raj Inter College,
   Bettiah, Bihar.

2. Sri Dwarika Prasad
   Govt. Inter College
   Kumarbagh,
   West Champaran, Bihar.

3. Sri M.K.Mukherjee
   Assistant Master,
   Ballygunja Govt.School,
   23/2 Beltola Road,
   Calcutta - 700020

4. Sri Nayan R.Basu
   Assistant Master,
   Hindu School,
   1B, B.C. Street
   Calcutta-700073

5. Sri P.K.Haldar,
   Assistant Master,
   Uttarpara Govt.School,
   P.O. Uttarpara
   Dist:- Hooghly, W.B.
   Pin-712258

6. Sri V.Murthy,
   Principal,
   K.V.S. Number 1,
   Bhubaneswar.

7. Sri V.S.Murthy
   PGT in Mathematics,
   K.V.S. No.1,
   Bhubaneswar.

8. Sri H.K.Padhi,
   PGT in Mathematics,
   K.V.S. No. 1,
   Bhubaneswar.

RESOURCE PERSON

1. Dr.K.K.Chakravarti,
   Reader & Prog.Director
   RIE, Bhubaneswar.

2. Dr.P.S.Tripathy,
   Reader in Mathematics,
   RIE, Bhubaneswar.

PERMUTATIONS AND COMBINATIONS

1. Introduction:

1.1 These two concepts are very familiar to both teachers and students alike as they have been set forth for generations in text books on elementary algebra. The emphasis here will be on methods of reasoning which can be used later and on the introduction of necessary concepts and working tools. Concepts of generating functions will be introduced here to show that they can be employed to obtain the results of both permutations and combinations in great generality.

Most of the proofs in permutations and combinations employ either or both of the following rules in one form or another.

Rule of Sum: If an object A may be chosen in m ways, and B in another n ways, then either A or B can be chosen in m + n ways.

Rule of product: If an object A may be chosen in m ways and thereafter B in n ways then A and B may be chosen in this order (AB) in m n ways. Notice that, in the first, the choices of A and B are mutually exclusive. The rule of product is used most often in cases where the order of choice is immaterial, that is, where the choices are independent. But the possibility of choices are also to be found out where they are not independent.

1.2 The basic definitions of permutations and combinations are given below.

__Definition:__ Permutations of n things taken r at a time or an r - permutation of n things is an ordered selection or arrangement of r of them.

Definition, combinations of n things taken r at a time or an r - combination of n things is a selection of r of them without regard to order.

In the definition of permutations, the meaning of ordered is that two arrangements are regarded as different if the order of selection is different even when the same things are selected. For example the 2 - permutations of 3 things a, b, c, are:

a b	a c	b c
b a	c a	c b

The 2-combinations of 3 things are ab, ac, bc only.

Thus the r- permutations may be regarded as made in two steps:

First: the selection of all possible sets of r things out of n things.

Second: the ordering of each of these in all possible ways.

In introducing the theory the n things taken are all unlike to make it simpler. In the most general case the things may be of K kinds with n_j things of the j th kind such that

$$n = n_1 + n_2 + \text{-----} + n_k$$

For example there may be 18 things of 4 kinds such that 3 is of the first kind, 4 of second kind, 5 of the third kind and 6 of the fourth kind.

$$18 = 3 + 4 + 5 + 6$$

Here $n = 18$, $K = 4$, $n_1 = 3$, $n_2 = 4$, $n_3 = 5$, $n_4 = 6$

1.2 r - permutations of n distinct objects.

1.2.1 Consider an r - permutation of n things or what we call one permutation or arrangement of n things taken r at a time. The first of the members in the arrangement can be chosen in n ways since n things are distinct. This done, the second can be chosen in $n - 1$ ways, and so on until the r th is chosen in $n - r + 1$ ways. By applying the rule of product repeatedly the number of r - permutations of n distinct things is :

$$P(n, r) = n(n-1)\cdots(n-r+1), \; n \geq r \quad \ldots(1)$$

If $r = n$ then

$$P(n, n) = n(n-1) \ldots\ldots 1 = n! \quad \ldots\ldots(2)$$

$P(n, 0)$ has no meaning combinatorially but is taken as 1 by convention.

Using (2), (1) may be written as

$$P(n, r) = \frac{n(n-1) \ldots (n-r+1)\,(n-r)(n-r-1) \ldots 1}{(n-r)(n-r-1) \ldots\ldots 1}$$

$$= \frac{n!}{(n-r)!} = \frac{P(n, n)}{P(n-r, n-r)}$$

$$P(n, n) = P(n, r)\, P(n-r, n-r) \quad \ldots\ldots(3)$$

This also follows from the fact that $n = n - r + r$, Again consider the r-permutations. They may contain a particular thing or may not. If they do not, the number is $P(n-1, r)$. If they do then in each $(r - 1)$ - permutations of $(n-1)$ things there are r positions in which the particular thing may appear. Therefore,

$$P(n, r) = P(n-1, r) + r\,P(n-1, r-1) \text{ ------- } (4)$$

The 24 3-permutations of 4 objects a,b,c, d are

a b c	a c d	b a c	b c a	c a b	c b a
a b d	a d b	b a d	b d a	d a b	d b a
a c d	a d c	c a d	c d a	d a c	d c a
b c d	b d c	c b d	c d b	d b c	d c b

2.2 The number of permutations of n things, p of which are of one kind, q of another, s of another and so on.

The things are not all unlike. So the number of the required permutations cannot be $P(n, n)$. Let x be the required number of permutations. Then $x \neq P(n, n)$.

Consider one such permutation. In this arrangement replace the p like things by p new things which are distinct from each other and from all other kinds of things given for arrangement. These p new things can be arranged among themselves in $p!$ ways while they keep occupying the old positions held by the p like things. Thus from one permutation of the problem $p!$ permutation can be generated where p like things are replaced

by p unlike things. The same logic can be repeated for every other set of like things. In the process of replacing the like objects we get

$$p + q + s + \text{-----} = n \text{ unlike objects.}$$

If x be the number of permutations under the question then

$$x\,(p\,!)\,(q\,!)\,(s\,!)\text{------} = n\,!$$

$$\text{or } x = \frac{n\,!}{p\,!\,q\,!\,r\,!\text{----}},\quad p + q + r + \text{--} = n \qquad \text{------(5)}$$

1.2.3 r - permutations with unrestricted repitition

Each place in an r-permutation may be filled in n different ways because each thing can be repeated an unrestricted number of times. By the rule of product, the number of permutations in question, that is, the r-permutations with repitition of n things, is

$$U\,(n,\,r) = n^r \quad \text{------}\,(6)$$

<u>Example</u>: A tape with r holes as used in teletype and computing machinary can be punched in 2^r ways. Here there are two "things" that each hole is either punched or not punched. So far r holes on the tape it can be punched in 2^r ways.

2. <u>Combinations</u>

2.1 Combinations of n things taken r at a time or r-combinations of n distinct things is

nothing but the number of ways in which r things can be selected out of the n things. This also means how many such groups of r distinct elements can be formed out of n things.

Let $C(n, r)$ be the number of such r- combinations of n things. Consider just one of them. The r things in this selection may be (ordered) arranged in $r!$ ways and each of these form different arrangements. So each one of them is an (ordered) r - permutation. So from one combination one can get $r!$ different arrangements of n things taken r at a time. Hence from $C(n, r)$ combinations one can get $r!\ C(n, r)$ different arrangements. But all of them together give the number of r- permutations of n things. So

$$r!\ C(n, r) = P(n, r) = n(n-1)\ (n-1) - - - (n-r+1) = \frac{n!}{(n-r)!} \quad \ldots\ldots\ldots(6)$$

$$C(n, r) = \frac{n!}{r!\ (n-r)!} = {}^{n}C = \binom{n}{r} \ldots(7)$$

$$\text{Also } C(n, n-r) = \frac{n!}{(n-r)!\ r!} = {}^{n}C_{n-r} = \binom{n}{n-r}$$

$$C(n, r) = C(n, n-r) \quad - - - - - -(8)$$

$C(n, o)$, the number of combinations of n things taken zero at a time has no combinatorial meaning. By (6) $C(n, 0) = 1$,

The values

$$C(n, r) = 0 \quad r < 0 \text{ and } r > n$$

are in agreement with (6).

C (-n, r) has no combinatorial meaning.

Algebraically $C(-n, r) = \binom{-n}{r} = \frac{(-n)(-n-1) \cdots (-n-r+1)}{r!}$

$$= (-1)^r \frac{(n+r-1)(n+r-2)\ldots(n+1)(n)}{r!}$$

$$= \frac{(-1)^r (n+r-1) \ldots\ldots (n+1)\, n\, (n-1) \ldots}{r!\,(n-1)(n-2) \ldots\ldots} \quad 2.1$$

$$= (-1)^r \frac{(n+r-1)!}{r!\,(n-1)!}$$

$$= (-1)^r C(n+r-1, n-1)$$

$$= (-1)^r C(n+r-1, r) \text{ by } \quad (8)$$

$$= (-1)^r \binom{n+r-1}{r}$$

Example: The six combinations of 4 distinct objects taken two at a time (n = 4, r = 2), labeling the objects a, b, c and d are: ab, ac, ad, bc, bd and cd .

2.2 Another derivation of (6) using the process of recurrence and the rule of sum :

The combinations or selections may be divided into those which include a particular thing and those which do not. The number of those of the first kind is C (n-1, r - 1), since occurrence of the particular one in each selection reduces both n and r by one. The number of those of the second kind is C(n-1, r) as the particular thing is never selected and hence groups of r are to be selected from the remaining (n-1) things. Hence

$$C(n, r) = C(n-1, r-1) + C(n-1, r), \; n \geqslant r \quad \ldots (8)$$

$$= C(n-1\; r-1) + C(n-2, r-1) + C(n-2, r)$$

$$= C(n-1, r-1) + C(n-2, r-1) + C(n-3, r-1)$$

$$+ C(n-3, r)$$

$$= C(n-1, r-1) + C(n-2, r-1) + C(n-3, r-1)$$

$$- - - + C(r-1, r-1) + C(r-1, r)$$

$$= C(n-1, r-1) + C(n-2, r-1) + - - - - +$$

$$C(r-1, r-1) \quad - - - - - - (9)$$

as $C(r-1, r) = 0$

Again from (7) $C(n, r) = C(n-1, r) + C(n-1, r-1)$
r - 1)

$$= C(n-1, r) + C(n-2, r-1) + C(n-2, r-2)$$

$$= C(n-1, r) + C(n-2, r-1) + C(n-3, r-2) +$$

$$C(n-3, r-3)$$

$$= C(n-1, r) + C(n-2, r-1)$$

$$+ C(n-3, r-2) + C(n-4, r-3)$$

$$+ C(n-4, r-4)$$

$$= C(n-1, r) + C(n-2, r-1) + C(n-3, r-2)$$

$$+ \ldots + C(n-1-r, 0) \quad - - - - - (10)$$

by repeated applications of (7).

2.3 A short table of $C(n, r)$ in the form of a triangle called Pascal Triangle is given below. Observe how lower row elements are filled in from the preceeding upper row with the help of (7).

Also remember C(n, r) = C(n, n - r) .

Numbers C(n , r)

3. Combinations with repetitions

3.1 We want to find the number of combinations of n distinct things take r at a time in which each of the things can appear any number of times, that is, 0 to r times. This is a function of both n and r. We denote this function by f (n, r).

The solution process just depends on the recurrence and the rule of sum. Suppose the things are numbered 1 to n. Then consider the selections which either contain 1 or they do not. If they do, they may contain it once, twice and so on upto r times. Since each of them contains 1 the number of such combinations or selections is f(n, r - 1), If the number of other selections which do not contain 1 in f(n - 1, r).

Therefore

$$f (n, r) = f(n, r-1) + f (n-1, r) \quad ..(11)$$

C (n, r)

r n	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

If $r = 1$, no repetitions of the things are possible.

Hence $f(n, 1) = n$ From (11)

$$f(n, 1) = f(n, 0) + f(n-1, 1)$$

$$f(n, 1) - f(n-1, 1) = f(n, 0)$$

$$\text{or } n - (n-1) = f(n, 0), \text{ or } 1 = f(n, 0)$$

If $n = 1$, only one combination is possible whatever r may be, so $f(1, r) = 1$.

From (1)

$$\begin{aligned}
f(n, 2) &= f(n, 1) + f(n-1, 2) \\
&= f(n, 1) + f(n-1, 1) + f(n-2, 2) \\
&= f(n,1) + f(n-1, 1) + f(n-2, 1) + f(n-3, 2) \\
&= f(n, 1) + f(n-1, 1) + f(n-2,1) + f(n-3, 1) + \ldots + f(2,1) + f(1,2) \\
&= n + (n-1) + (n-2) + \ldots + 2 + 1 \\
&= \frac{n(n+1)}{2} = \binom{n+1}{2} = C(n+1, 2)
\end{aligned}$$

Alternatively $n + (n-1) + (n-2) + \ldots + 2 + 1$

$$\begin{aligned}
&= C(n,1) + C(n-1, 1) + C(n-2, 1) + \ldots + C(2,1) + C(1, 1). \\
&= C(n+1, 2) \text{ by (9)} \\
&= \binom{n+1}{2}
\end{aligned}$$

Similarly

$$\begin{aligned}
f(n, 3) &= f(n,2) + f(n-1, 3) \\
&= f(n,2) + f(n-1,2) + f(n-2, 3)
\end{aligned}$$

$= f(n, 2) + f(n-1, 2) + f(n-2, 2) + f(n-3, 3)$

$= f(n, 2) + f(n-1, 2) + f(n-2, 2) + f(n-3, 2)$

$+ f(2, 2) + f(1, 3)$

by repeated application of (11)

$= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \ldots + \binom{3}{2} + 1$

$= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \ldots + \binom{3}{2} + \binom{2}{2}$

$= C(n+1, 2) + C(n, 2) + C(n-1, 2) + \ldots + C(3, 2)$

$+ C(2, 2)$

$= C(n+2, 3)$ by (11)

$= \binom{n+2}{3}$

From this the number of r-combinations of n distinct things with repetitions upto to r times for each is

$$f(n, r) = \binom{n+r-1}{r} \quad \ldots\ldots\ldots (12)$$

Also $f(n, r-1) + f(n-1, r)$

$= \binom{n+r-2}{r-1} + \binom{n+r-2}{r} = C(n+r-2, r-1)$

$+ C(n+r-2, r)$

$= C(n+r-1, r)$ by (8)

$= f(n, r)$

3.2 $f(n, r) = \binom{n+r-1}{r}$ can be proved by induction for all n and r.

Proff. Let $f(k, s) = \binom{k+s-1}{s}$ $k \leq n$

$s \leq r$.

by induction assumption. Also $f(n,3) = \binom{n+2}{3}$

We claim $f(n, r+1) = \binom{n+r}{r+1}$ and $f(n+1, r) = \binom{n+r}{r}$

Now $f(n, r+1) = f(n, r) + f(n-1, r+1)$

$= f(n,r) + f(n-1,r) + f(n-2, r+1)$

$= f(n,r) + f(n-1, r) + f(n-2,r) + f(n-3, r) + \ldots$

$+ f(r, r) + f(r-1, r) + \ldots + f(1,r+1)$ by (1_1)

$= \binom{n+r-1}{r} + \binom{n+r-2}{r} + \ldots + \binom{r+r-1}{r} + \binom{r+r-2}{r}$

$+ \quad + \binom{r+1}{r} + 1$

as $f(1, r+1) = 1$

$= \binom{r}{r}$

$= \binom{n+r-1}{r} + \binom{n+r-2}{r} + \ldots +$

$\binom{2r-1}{r} + \binom{2r-2}{r}$

$+ \quad + \binom{r+1}{r} + \binom{r}{r}$

by replacing 1 by $\binom{r}{r}$

$= C(n+r-1, r) + C(n+r-2, r) + \quad +$

$C(r, r)$

$= C(n+r, r+1)$

$= \binom{n+r}{r+1} = \binom{n+r+1-1}{r+1}$

$$f(n+1, r) = f(n+1, r-1) + f(n, r)$$

$$= f(n+1, r-1) + f(n, r-1) + f(n-1, r)$$

$$= f(n+1, r-1) + f(n, r-1) + f(n-1, r-1)$$

$$+ \dots + f(r-1, r-1) + \dots + f(1, r-1)$$

$$= \binom{n+r-1}{r-1} + \binom{n+r-2}{r-1} + \dots +$$

$$\binom{2r-1}{r-1} + \dots + \binom{r-1}{r-1}$$

by replacing $f(1, r-1) = 1 = \binom{r-1}{r-1}$

$$= C(n+r-1, r-1) + C(n+r-2, r-1)$$

$$+ \dots + C(r-1, r-1)$$

$$= C(n+r, r) = \binom{n+r}{r}$$

$$= \binom{n+1+r-1}{r}$$

Hence by induction

$$f(n, r) = \binom{n+r-1}{r} \text{ for all } n \text{ and } r$$

Example: The 3-combination of 5 things 1,2,3,4,5

with repetitions is $\binom{5+3-1}{3} = \binom{7}{3}$

$$= C(7, 3)$$

$$= \frac{7!}{3!\,4!} = \frac{5.\ 6.\ 7}{\cancel{1}.\ \cancel{2}.\ \cancel{3}} = 35$$

The selection are

111, 112, 113, 114, 115, 123, 124, 125, 134,
135, 145, 222, 221, 223, 224, 225, 234, 235
245, 333, 331, 332, 334, 335, 345, 444, 441,
442, 443, 445, 555, 551, 552, 553. and 554.

They form 35 selections with repetitions of each of the objects 0 to 3 times.

3.3 The result (12) $f(n,r) = \binom{n+r-1}{r}$ can also beproved simply for which argument was forwarded by none other than Euler himself. Consider one r-combination, say

$$C_1\ C_2\ C_3\ C_4\ \text{------}\ C_r$$

in rising order (with like elements taken to be rising), of n things numbered 1, 2, 3, ..., n, where repetitions were allowed. Because of unlimited repetition from 0 to r times, any number of C's may be alike. From this r - combination from a set

$$d_1\ d_2\ \text{-----}\ d\ r$$

by the rules

$$d_1 = C_1 + 0$$

$$d_2 = C_2 + 1$$

$$d_3 = C_3 + 2$$

$$d_r = C_r + (r - 1)$$

Thus d's are all unlike in $d_1\ d_2 \ldots d\ r$. The number of d's is the same as that of C'r in each selection.

Each r-combination of C's produces an r - combination of distinct d's. The number of d's so formed will be n + r - 1 numbered from 1 to n + r - 1. Each r-combination of d's are selections of r distinct elements from n + r - 1 things numbered 1 to n + r - 1. Hence the number of such selections is C (n + r - 1, r).

Hence,

$$f(n, r) = C(n + r - 1, r) = \binom{n + r - 1}{r}$$

which is in complete agreement with (12).

4. Generating Function for Combinations

4.1 The calculations given above can be unified and generalised by a simple mathematical procedure called the generating function.

Consider three objects x_1, x_2 and x_3. Then the algebraic product

$$(1 + x_1 t)(1 + x_2 t)(1 + x_3 t)$$

$$= 1 + (x_1 + x_2 + x_3) t + (x_1 x_2 + x_2 x_3 + x_3 x_1) t^2 + x_1 x_2 x_3 t^3 \text{ -------(13)}$$

When multiplied and arranged in terms of powers of t.

$$= 1 + a_1 t + a_2 t_2 + a_3 t^3$$

Where $a_1 = x_1 + x_2 + x_3$

$a_2 = x_1 x_2 + x_2 x_3 + x_3 x_1$

$a_3 = x_1 x_2 x_3$

Here a_1, a_2, a_3 are each functions of x_1, x_2, x_3. Infact they are called symmetric functions of the variables x_1, x_2 and x_3.

Also (13) can be written as

$$(1 + x_1 t)(1 + x_2 t)(1 + x_3 t) = \sum_{r=0}^{3} a_r (x_1, x_2, x_3) t^r \text{ ----------- (14)}$$

where $a_0 = 1$

$a_1 = x_1 + x_2 + x_3$

$a_2 = x_1 x_2 + x_2 x_3 + x_3 x_1$

$a_3 = x_1 x_2 x_3$, It is observed that a_1 contains one term for each combination of 3 things x_1, x_2 and x_3 taken 1 at a time or 1 - combination of 3 things. a_2 contains one term for each combination of 3 things x_1, x_2, and x_3 taken 2 at a time or 2 - combination of 3 things.

a_3 contains one term meaning combination of 3 things taken 3 at a time. Henee the number of such combination or selection is obtained by putting

$$x_1 = 1,\ x_2 = 1,\ x_3 = 1 \text{ in } \quad (13)$$

Then

$$(1 + t^3) = 1 + 3t + 3\,t^2 + t^3$$

$$= \sum_{r=0}^{3} a_r\,(1, 1, 1)\,t^r$$

$$a_r\,(1,1,1) = C\,(3, r),\ r = 0, 1, 2, 3$$

$$a_0 = 1 = C\,(3, 0) = 1,\ a_1 = C(3,1) = 3,$$

$$a_2 = C\,(3, 2) = 3$$

$$\text{and } a_3 = C\,(3, 3) = 1 .$$

In Case of n distinct things labelled x_1 to x_n, it is clear that

$$(1 + x_1 t)(1 + x_2 t) \ldots\ldots (1 + x_n t)$$

$$= 1 + a_1 (x_1, x_2, \ldots x_n)\, t + a_2 (x_1, x_2 \ldots, x_n)\, t^2 + \ldots + a_r (x_1, x_2, \ldots, t_n)\, t^r + \ldots + a_n (x_1, x_2, \ldots, x_n)\, t^n$$

$\therefore$ $(1 + t)^n = 1 + a_1 (1,1, .., 1) t + a_2 (1, 1,...,1) t^2 + ... + a_r (1, 1, ...1) t^r + ... + a_n (1,1,...1)$

$$= \sum_{r=0}^{n} a_r (1, 1, ..., 1) t^r$$

$$= \sum_{r=0}^{n} C (n, r) t^r \quad .... (15)$$

Where C (n, r) stands for the usual binomial coefficients, that is the coefficients in the expansion of $(a + b)^n$.

The expression $(1 + t)^n$ is called the enumerating generating function or simply the enumerator of combinations of n distinct things.

4.2 The applications of the enumerator is indicated in the following examples.

<u>Example-1</u>. In equation (15) put t = 1 then

$$2^n = \sum_{r=0}^{n} C (n, r) = \sum_{r=0}^{n} \binom{n}{r} \quad ....(16)$$

That is the total number of combination n distinct things , taken any number at a time, is 2^n .

The total number of combinations of n things is 2^n because in this total count each thing either appears in a combination or it does not.

With t = - 1 in (15)

$$0 = \sum_{r=0}^{n} (-1)^r C (n, r) = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$$

$$0 = 1 - \binom{N}{1} + \binom{n}{2} - \binom{n}{3} + ... + (-1)^n \binom{n}{n} \quad ......(17)$$

Adding and subtracting (16) and (17) we get

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\ldots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\ldots$$

$$= 2^{n-1}$$

or

$$\sum_{r=0}^{n} \binom{n}{2r+1} = \sum_{r=0}^{n} \binom{n}{2r} = 2^{n-1}$$

Example-2: Writing $n = n - m + m$, we have

$$(1+t)^n = (1+t)^{n-m} (1+t)^m$$

Now equating the coefficients of t^r on both sides, we have $C(n, r) = C(n-m, 0)\ C(m,r) + C(n-m, 1)\ C(m, r-1) + \ldots + C(n-m, r)\, C(m, 0)$.

$$\text{or } \binom{n}{r} = \binom{n-m}{0}\binom{m}{r} + \binom{n-m}{1}\binom{m}{r-1} + \ldots\ldots \binom{n-m}{r}\binom{m}{0}$$

4.3 The result (15) is only a beginning, A question naturally arises:

What are the generating functions and enumerators when the elements to be combined are not distinct ?

In the expression $(1 + x_1 t)(1 + x_2 t) \ldots (1 + x_n t)$ each factor of the product is a binomial (expression containing two terms) which indicates in the terms 1 and $x_k t$ the fact

that the element x_k may not or may appear in a given combination. The product generates combinations because the coefficient of t^r is obtained by picking unity terms (one) from (n - r) factors and terms like x_k t from the r remaining factors in all possible ways. These are the combinations or selections of n things take r at a time or r - combinations of n things. The factors are limited to two terms because no object can appear more than once in any combination or selection.

Hence, if the combinations can have the object x_k 0, 1, 2, ... j times then the generating function is altered by

$$1 + x_k t + x_k^2 t^2 + ... + x_k^j t^j$$

in place of $1 + x_k t$ in (13)

Also the factors may be specified quite logically. If x_k is always to appear an even number of times, but not more than j times with $j \leqq 2i + 1$ then the factor is altered by

$$1 + x_k^2 t^2 + x_k^4 t^4 + ... + y_k^{2i} t^{2i}$$

in place of $1 + y_k t$.

Therefore the generating function describes not only the kinds of things but also the kinds of combinations. A factor x_k^i in any term of a coefficient of a power of t in the generating function indicates that the thing x_k appears i times in the corresponding combination.

Example-1 For combinations with unlimited repetition of things of n kinds, and no restriction on the number of times any such thing can appear, the enumerating generating function is

$$(1 + t + t^2 + t^3 + \ldots\ldots + t^k + \ldots\ldots)^n \quad \ldots\ldots(18)$$

But it is known that

$$1 + t + t^2 + t^3 + \ldots\ldots + t^k + \ldots = (1-t)^{-1}$$

Therefore,

$$(1 + t + t^2 + \ldots\ldots + t^k + \ldots\ldots)^n = (1-t)^{-n}$$

Again $(1-t)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (- t)^r$, n a positive integer.

$$\binom{-n}{r} = \frac{(-n)\ (-n-1)\ \text{------}(-n-r + 1)}{r\,!}$$

$$= \frac{(-1)^r\ n\ (n + 1)\ \ldots\ldots\ (n + r - 1)}{r\,!}$$

$$= \frac{(-1)^r\ (n + r -1)\ (n + r -2)\ \ldots(n+1)\ n}{r\,!}$$

$$= \frac{(-1)^r\ (n + r - 1)!}{r\,!\ (n - 1)\,!}$$

$$= (- 1)^r \binom{n + r - 1}{r}$$

Hence,

$$(1 + t + t^2 + \ldots\ldots + t^k + \ldots)^n = (1-t)^{-n}$$

$$= \sum_{r=0}^{\infty} \binom{-n}{r} (-t)^r$$

$$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} (-1)^r,\ (-t)^r$$

$$= \sum_{r=0}^{\infty} \binom{n + r - 1}{r} t^r$$

These result conforms to the result obtained in (12) . Utilising the notations from (12) we have

$$(1 + t + t^2 + \ldots + t^k + \ldots)^n = (1 - t)^{-n}$$

$$= \sum_{r=0}^{\infty} \binom{n + r - 1}{r} t^r$$

$$= \sum_{r=0}^{\infty} f(n, r) t^r \quad \text{----} \quad (19)$$

Example-2 If one puts the further condition that at least one object of each kind must appear, the enumerating generating function, then takes the form

$$(t + t^2 + \text{---} + t^k + \text{-----})^n$$

$$= \left\{ t \, (1 + t + \text{----} + t^{k-1} + \text{----}) \right\}^n$$

$$= t^n \, (1 + t + \text{----} + t^{k-1} + \text{----})^n$$

$$= t^n \left((1 - t)^{-1} \right)^n = t^n \, (1 - t)^{-n}$$

$$= t^n, \sum_{r=0}^{\infty} \binom{n + r - 1}{r} t^r \quad \text{from (19)}$$

$$= t^n \left\{ \binom{n-1}{0} t^0 + \binom{n}{1} t + \binom{n+1}{2} t^2 + \ldots + \binom{n + r - 1}{r} t^r + \text{------} \right\}$$

$$= \binom{n-1}{0} t^n + \binom{n}{1} t^{n+1} + \binom{n+1}{2} t^{n+2} + \text{-------} + \binom{n + r - 1}{r} t^{n+r} +$$

Writing $n + r = m$ and replacing the variable r by $m - n$ in the general term, we get

$$= \sum_{m=n} \binom{m-1}{m-n} t^m \text{ - - - - - - -}$$

$$= \sum_{m=n} \binom{m-1}{n-1} t^m \text{ - - - - - - - - -} (20)$$

The number of combinations in question is therefore

$$0 \text{ for } m < n$$

$$\binom{m-1}{n-1} = C(m-1, n-1) \text{ for } m \geq n$$

Example-3 For $n = 4$ and elements a, b, c, d the number of 4.- combinations will be the coefficient of t^4 i, e, $\binom{r-1}{n-1}$ where $r = 4$

$n = 4$

Which is $\binom{3}{3} = C(3, 3) = 1$ and this is abcd.

The number of 6-combinations is the coefficient of t^6 which is $\binom{6-1}{4-1}$

$= \binom{5}{3} = C(5, 3) = \frac{5!}{3!\ 2!} = 10$. The 10

6 - combinations are

a a a b c d

a a b b c d

a a b c c d

a a b c d d

a b b b c d

a b b c c d

a b b c d d

a b c c c d

a b c d d d

a b c c d d

Example-4 For combinations with unlimited repetition of objects of n kinds subject to the condition that each object can appear an even number of times unlimitedly, the enumerator

$$(1 + t^2 + t^4 + \cdots + t^{2k} + \cdots)^n$$

$$= \left((1 + t^2)^{-1} \right)^n$$

$$= (1 + t^2)^{-n}$$

$$= \sum_{r=0}^{\infty} \binom{-n}{r} (- t^2)^r$$

$$= \sum_{r=0}^{\infty} \binom{n + r - 1}{r} t^{2r} \quad \text{-------} (21)$$

by applying (19).

Thus the r-combinations for r odd is 0. The 2r - combinations are the same as the r - combinations in (19).

Now $(1-t^2)^{-n} = (1-t)^{-n} (1 + t)^{-n}$

$$= \left\{ \sum_{r=0}^{\infty} \binom{n + r - 1}{r} t^r \right\} \left\{ \sum_{K=0}^{\infty} \binom{n+K-1}{k} t(- 1)^k t^k \right\}$$

Expanding the summations

$$(1-t^2)^{-n} = \Big\{ 1 + \binom{n}{1} t + \binom{n+1}{2} t^2 + \binom{n+2}{3} t^3 + \binom{n+3}{4} t^4 + \binom{n+67}{5} t^5 + \ldots \Big\} x$$

$$\Big\{ 1 - \binom{n}{1} t + \binom{n+1}{2} t^2 - \binom{n+2}{3} t^3 + \binom{n+3}{4} t^4 - \binom{n+4}{5} t * \Big\}$$

Collecting the coefficients of t, t^3, t^5 -----

$$\binom{n}{1} - \binom{n}{1} = 0$$

$$\binom{n+2}{3} - \binom{n+1}{2}\binom{n}{1} + \binom{n}{1}\binom{n+1}{2} - \binom{n+2}{3} = 0$$

$$\binom{n+4}{5} - \binom{n+3}{4}\binom{n}{1} + \binom{n+2}{3}\binom{n+1}{2} - \binom{n+1}{2}\binom{n+2}{3} + \binom{n}{1}\binom{n+3}{4} - \binom{n+4}{5} = 0$$

$$\ldots\ldots(23)$$

as no object can appear an odd number of times in any combination.

Consider r to be an odd integer. Then from (23) one can have

$$\binom{n+r-1}{r} - \binom{n+r-2}{r-1}\binom{n}{1} + \binom{n+r-3}{r-2} \times \binom{n+1}{2}$$

$$- - - - - - - - + (-1)^k \binom{n+r-k-1}{r-k} \cdot \binom{n+k-1}{k} +$$

$$(-1)^r \binom{n+r-1}{r} = 0$$

Consider r to be an even integer. Let $r = 2s$

The coefficient of $t^r = t^{2s}$ from both sides of (2222) when $s = 0, 1, 2, 3, - - - -$

$s = 0$ $\quad 1 = 1$

$$s = 1 : \binom{n}{1} = \binom{n+1}{2} - \binom{n}{1}\binom{n}{1} + \binom{n+1}{2}$$

$$s = 2 : \binom{n+1}{2} = \binom{n+3}{4} - \binom{n+2}{3}\binom{n}{1} + \binom{n+1}{2}\binom{n+1}{2} - \binom{n}{1}\binom{n+2}{3} + \binom{n+3}{4}$$

$$= 3; \binom{n+2}{3} = \binom{n+5}{6} - \binom{n+4}{5}\binom{n}{1} + \binom{n+3}{4}\binom{n+1}{2} -$$

$$= \binom{n+2}{3}\binom{n+2}{3} + \binom{n+1}{2}\binom{n+3}{4} - \binom{n}{1}\binom{n+4}{5} + \binom{n+5}{6} ---- (24)$$

From (24), one can draw

$$\binom{n+s-1}{s} = \binom{n+2s-1}{2s} - \binom{n+2s-2}{2s-1}\binom{n}{1} + \binom{n+2s-3}{2s-2}\binom{n+1}{2} ----+$$

$$+ \binom{n+2s-k-1}{2s-k}\binom{n+k-1}{k}(-1)^k$$

$$+ --- + \binom{n+2s-1}{2s}$$

$$\binom{n+s-1}{s} = \sum_{k=0}^{2s} (-1)^k \binom{n+2s-K-1}{2s-k} \times \binom{n+k-1}{k}$$

Examples like these can be multiplied indefinitely. The most important thing is to observe the following:

In forming a combination, the objects are chosen independently. The generating function takes advantage of this independence by a rule of multiplication. In reality, each factor in the product is a generating function for the objects of the given kind.

5. Generating Functions for Permutations.

5.1 It is difficult to provide a generating function which will exhibit and stand for r-permutations of n things since the different arrangements $x_1 x_2$ and $x_2 x_1$ are same in our algebraic system. Neverthe less, the enumerators are easy to find. For n unlike things it follows that

$$(1 + t)^n = \sum_{0}^{n} P(n, r) \frac{t^r}{r!} \quad \ldots\ldots(25)$$

at once from (1). Thus $P(n, r)$ is the coefficient of $\frac{t^r}{r!}$ in the expansion of $(1 + t)^n$.

This paves the way for generalisation. If an element can appear $0, 1, 2 - - - - K$ times, or if there are K elements of a given kind, a factor $(1 + t)$ on the left of (25) is replaced by

$$1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + - - - - + \frac{t^k}{k!}$$

This is because the number of permutations of n things, p of which are of one kind, q of another and so on, is given by (5) as

$$\frac{n!}{p!\,q!}$$

which is the coefficient of $\frac{t^n}{n!}$ in the product.

$$\frac{t^p}{p!} \frac{t^q}{q!} - - - - - - -, \; p + q + - - = n.$$

This corresponds to the fact that the things of the first kind appear exactly p times, second kind appear exactly q times and so on.

5.2 If the permutations are governed by the conditions that the first of n elements is to appear

$$X_0(1),\ X_1(1),\ \text{- times}$$

the second of n elements is to appear

$$X_0(2), X_1(2), X_2(2), \text{- times}$$

the k th of n elements is to appear

$$X_0(k),\ X_1(k),\ \text{- - - - - times}$$

the last of n elements is to appear

$$X_0(n),\ X_1(n),\ \text{- - - - - - times}$$

then the number of permutations of n things taken r at a time or r-permutations is the coefficient of $\frac{t^r}{r!}$ in the product:

$$\left(\frac{t^{X_0(1)}}{(X_0(1))!} + \frac{t^{X_1(1)}}{(X_1(1))!} + \cdots\right)\left(\frac{t^{X_0(2)}}{(X_0(2))!} + \frac{t^{X_1(2)}}{(X_1(2))!} + \cdots\right)$$

$$\cdot\left(\frac{t^{X_0(n)}}{(X_0(n))!} + \frac{t^{X_1(n)}}{(X_1(n))!} + \cdots\right)$$

$$= \prod_{K=1}^{n}\left(\frac{t^{X_0(K)}}{(X_0(K))!} + \frac{t^{X_1(K)}}{(X_1(K))!} + \frac{t^{X_2(K)}}{(X_2(K))!} + \cdots\right)$$

This generating function is called an expontial generating function as .

$$e^{at} = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \cdots + \frac{(at)^r}{r!} + \cdots$$

$$= \sum_{r=0}^{\infty} a^r \frac{t^r}{r!}$$

5.3 For r - permutations of n different things with unlimited repetitions (no restriction regarding the number of times a thing can appear), the enumerator is

$$(1 + t + \frac{t^2}{2\,!} + \frac{t^3}{3\,!} + - - - - - - -)^n$$

$$= (e^{t})^n = e^{nt} = \sum_{r=0}^{\infty} n^r, \frac{t^r}{r\,!}$$

Hence the number of r - permutations is n^r which is in agreement with (6).

5.4 For permutations of n objects subject to the condition that each object must appear at least once, the enumerator is

$$(t + \frac{t^2}{2!} + \frac{t^3}{3!} + ---)^n = (e^t - 1)^n$$

$$= \sum_{j=0}^{n} \binom{n}{j} e (n - j)^t, (-1)^j$$

$$\sum_{j=0}^{n} \binom{n}{j} e^{(n-j)t} (- 1)^j$$

$$= e^{nt} - \binom{n}{1} e^{(n-1)t} + \binom{n}{2} e^{(n-2)t} - \binom{n}{3}$$

$$e^{(n-3)\,t} + - - - - + (-1)^n \binom{n}{n} e^{(n - n)\,t}$$

$$= \left\{1 + nt + \frac{t^2 n^2}{2\,!} + \frac{n^3\, t^3}{3\,!} + - - -\right\} - \binom{n}{1}$$

$$\left\{ 1 + (n-1)\, t + \frac{(n-1)^2\, t^2}{2\,!}) + - -\right\} + - + (-1)^k$$

$$\binom{n}{k} \left\{1 + (n-k)t + \frac{(n-k)^2\, t^2}{2\,!} + - - - \right\}$$

$$+ - - - + \left\{(-1)^n \binom{n}{n}\right\}\left\{1 + (n - n)\, t + \frac{(n-n)^2\, t^2}{2!} + \cdot\right.$$

Coefficient of $\frac{t^r}{r\,!}$ in this expansion is

$$n^r - \binom{n}{1}(n-1)^r + \binom{n}{2}(N-2)^r + ---$$

$$+ (-1)^K \binom{n}{k}(n-k)^r + ---- + (-1)^n \binom{n}{n}(+0)$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j}(n-j)^r$$

$$\therefore (e^t - 1)^n = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left\{\sum_{j=0}^{n} (-1)^j \binom{n}{j}(n-j)^r\right\}$$

5.5 For r - permutations of n elements, p of which are of one kind, q of another and so on, the enumerating generating function is the product:

$$\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + --- + \frac{t^p}{p!}\right)$$

$$\left(1 + t + \frac{t^2}{2!} + ----- + \frac{t^q}{q!}\right) -----$$

$$= \prod \left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^K}{K!}\right) \quad \text{where } K = p, q, \cdots$$

Problems

1. (a) Show that p plus signs and q minus signs may be placed in a row so that no two minus signs are together in $\binom{p+1}{q}$ ways.

(b) Show that n signs, each of which may be plus or minus, may be placed in a row with no two minus signs together in f(n) ways, where $f(0) = 1$, $f(1) = 2$ and $f(n) = f(n-1) + f(n-2)$, $n > 1$

(c) Comparison of (a) and (b) requires that

$$f(h) = \sum_{q=0}^{m} \binom{n-q+1}{q} \quad \text{where} \quad m = \left[\frac{(n+1)}{2}\right]$$

with $[x]$ indicating the largest integer not greater than x. Show that

$$g(n) = \sum_{q=0}^{m} \binom{n-q+1}{q} = g(n-1) + g(n-2)$$

and $g(0) = 1$, $g(1) = 2$, so that $g(n) = f(n)$.
The numbers $f(n)$ are Fibonacci numbers.

Solution:

There are no restrictions on plus signs. So the minus signs should occupy places between two positive signs only. p plus signs create $p + 1$ places in between them to be occupied by q minus signs. So q places are to be selected out of the $(p + 1)$ positions to be occupied by q minus signs. And this can be accomplished in

$\binom{p+1}{q}$ ways.

Also $q \leq p + 1$. Otherwise two minus signs will be together in a row.

(b) The number of ways of puttin, a total of n signs both positive and negative in a row in which no two minus signs are together is $f(n)$ Similarly $f(n-1)$ and $f(n-2)$ are defined.

If to each of the $f(n-1)$ arrangements of a total of $(n-1)$ signs, both plus and minus one plus sign is added then it becomes an arrangement of n signs in a row where no two minus signs come together. So each one of $f(n-1)$ gives rise to an arrangement of $f(n)$. Now we consider the arrangements of $f(n-1)$ to which we can add minus sign to form a member of $f(n)$. Now we have to choose those members of $f(n-1)$ which does not have minus signs at the end. These must have plus signs at the end.

These must have plus signs at the
And therefore they must be members of $f(n-2)$ because the members of $f(n-1)$ with plus signs at the end were obtained from each member of $f(n-2)$.

Therefore $f(n) = f(n-1) + f(n-2)$

$$f(3) = f(2) + f(1)$$

$f(1)$ is given by $\pm$ $\therefore f(1) = 2$.

Considering $f(2)$ we can have

$$\begin{array}{cc} + & - \\ - & + \\ + & + \\ + & - \\ - & + \end{array}$$

and no other possibilities are there.

$$f(2) = 3$$

For $f(3)$

$$\begin{array}{ccc} + & + & + \\ + & - & + \\ - & + & + \\ + & + & - \\ - & + & - \end{array}$$

these are the possibilities. So

$$f(3) = 5$$

$$f(3) = f(2) + f(1)$$

$$5 = 3 + f(1) \qquad f(1) = 2$$

Again $f(2) = f(1) + f(0)$

$$3 = 2 + f(0) \qquad f(0) = 1$$

(c) From (a) above

$$f(p+q) = \binom{p+1}{q} \text{ where } q \leqq p+1$$

Now writing $p + q = n$

$$f(n) = \binom{n-q+1}{q} \text{ where } 2q \leqq p+q+1$$

$$q \leqq n+1$$

So for a particular choice of p and q with

$$p + q = n$$

$$f(n) = \binom{n-q+1}{q} \text{ where } q \leq n+1$$

q being a positive integer.

Now taking all possible choices of p and q with $p + q = n$, we have

$$f(n) = \sum_{q=0}^{m} \binom{n-q+1}{q} \text{ where } m = \left[\frac{(n+1)}{2}\right]$$

With $[x]$ is indicating the largest integer not greater than x.

$$\text{Let } g(n) = \sum_{q=0}^{m} \binom{n-q+1}{q},\ m = \left[\frac{(n+1)}{2}\right]$$

$$\text{Now } \binom{n-q+1}{q} = \binom{n-q}{q} + \binom{n-q}{q-1}$$

$$\text{So } g(n) = \sum_{q=0}^{m} \binom{n-q}{q} + \sum_{q=0}^{m} \binom{n-q}{q-1}$$

When n is even $m = \left[\frac{n+1}{2}\right] = \frac{n}{2}$

n - 1 is odd $m = \left[\frac{n-1+1}{2}\right] = \frac{n}{2}$

(n-2) is even $m = \left[\frac{n-2+1}{2}\right] = \left[\frac{n-1}{2}\right] = \frac{n-2}{2} = \frac{n}{2} - 1$

When n is odd $m = \left[\frac{(n+1)}{2}\right] = \frac{n+1}{2}$

n - 1 is even $m = \left[\frac{n-1+1}{2}\right] = \frac{n}{2}$

m - 2 is odd $m = \left[\frac{n-2+1}{2}\right] = \left[\frac{n-1}{2}\right] = \frac{n-1}{2}$

$$\text{Thus } \sum_{q=0}^{m} \binom{n-q}{q} = \sum_{q=0}^{m} \binom{(n-1)-q+1}{q} = g(n-1)$$

when n is even

$$\sum_{q=0}^{m} \binom{n-q}{q} = \sum_{q=0}^{n/2} \binom{(n-1)-q+1}{q} + \binom{n-\frac{n+1}{2}}{\frac{n+1}{2}}$$

$$= g(n-1) + \binom{\frac{n-1}{2}}{\frac{n+1}{2}}$$

$$= g(n-1) + 0 = g(n-1) \text{ when } n \text{ is odd}$$

$$\sum_{q=0}^{m} \binom{n-q}{q-1} = \binom{n}{-1} + \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-\frac{n}{2}}{\frac{n}{2}-1} \text{ when } n \text{ is even}$$

$$= 0 + \sum_{q=0}^{\frac{n}{2}-1} \binom{n-q-1}{q} \text{ when } n \text{ is even}$$

$$= \sum_{q=0}^{\frac{n}{2}-1} \binom{(n-2)-q+1}{q} = g(n-2) \text{ when } n \text{ is even}$$

$$\sum_{q=0}^{m} \binom{n-q}{q-1} = \binom{n}{-1} + \binom{n-1}{0} + \binom{n-2}{1} + \dots + \binom{n-\frac{n+1}{2}}{\frac{n+1}{2}-1}$$

$$= 0 + \sum_{q=0}^{\frac{n-1}{2}} \binom{n-q-1}{q}$$

$$= \sum_{q=0}^{\frac{n-1}{2}} \binom{(n-2)-q+1}{q}$$

$$= g(n-2) \text{ when } n \text{ is odd.}$$

Hence the solution to the problem.

2. Show that

a) $n \binom{n}{r} = (r+1)\binom{n}{r+1} + r\binom{n}{r}$

b) $\binom{n}{2}\binom{n}{r} = \binom{r+2}{2}\binom{n}{r+2} + 2\binom{r+1}{2}$

$$\binom{n}{r+1} + \binom{r}{2}\binom{n}{r}$$

c) $\binom{n}{s}\binom{n}{r} = \sum_{k=0}^{q} \binom{s}{k}\binom{r+s-k}{r-k}\binom{n}{r+s-k},$

$$q = \min(r, s)$$

<u>Solution:</u>

a) $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$ is true.

Now $\dfrac{\binom{n+1}{r+1}}{\binom{n}{r}} = \dfrac{\dfrac{(n+1)!}{(r+1)!\,(n-2)!}}{\dfrac{n!}{r!\,(n-r)!}} = \dfrac{n+1}{r+1}$ after cancellations.

$$(r+1)\binom{n+1}{r+1} = (n+1)\binom{n}{r}$$

Substituting again for $\binom{n+1}{r+1}$

$$(r+1)\left\{\binom{n}{r} + \binom{n}{r+1}\right\} = n\binom{n}{r} + \binom{n}{r}$$

$$r\binom{n}{r} + (r+1)\binom{n}{r+1} = n\binom{n}{r}.$$

b) Selections of K out of n and then 2 out of these K objects is given by

$$C(n, k),\ C(k, 2) = \binom{n}{k}\binom{k}{2}$$

It is the same as the selection of 2 out of n and selection of K-2 out of the remaining

(n-2) objects. This number is $\binom{n}{2} C(n-2,k-2)$

$$= \binom{n}{2}\binom{n-2}{k-2}$$

Hence $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$

Calculation from the formula

$$\binom{n}{k}\binom{k}{2} = \frac{n!}{k!\,(n-k)!} = \frac{k!}{(k-2)!\,2!}$$

$$= \frac{n!}{(n-2)!}$$

$$= \frac{n!}{2!\,(n-2)!} = \frac{(n-2)!}{(k-2)!\,.\,(n-k)!} = \binom{n}{2}\binom{n-2}{k-2}$$

Right hand side of (b)

$$\binom{r+2}{2}\binom{n}{r+2} + 2\binom{r+1}{2}\binom{n}{r+1} + \binom{r}{2}\binom{n}{r}$$

$$= \binom{n}{2}\binom{n-2}{r} + 2\binom{n}{2}\binom{n-2}{r-1} + \binom{n}{2}\binom{n-2}{r-2}$$

$$= \binom{n}{2}\left\{\binom{n-2}{r} + \binom{n-2}{r-1} + \binom{n-2}{r-1} + \binom{n-2}{r-2}\right\}$$

$$= \binom{n}{2}\left\{\binom{n-1}{r} + \binom{n-1}{r-1}\right\} = \binom{n}{2}\binom{n}{r}$$

c) Number of selections of (r + s - k) from n and r-k from these selected (r + s - k) objects is given by

$$\binom{n}{r+s-k}\,.\binom{r+s-k}{r-k}$$

Which is the same as the number of selections of r - k from n and (r + s - k) - (r-k) = s objects from n - (r - k)

$$\binom{n}{r+s-k}\binom{r+s-k}{r-k} = \binom{n}{r-k}\binom{n-r+k}{s}$$

$$= \binom{n}{n-r+k}\binom{n-r+k}{s}$$

c) $\sum_{k=0}^{n-m} (-1)^k \binom{n-k}{m} \binom{n}{k}$

$= \sum_{k=0}^{n-m} (-1)^k \binom{n-k}{m} \binom{n}{n-k}$

$= \sum_{k=0} (-1)^k \binom{n}{m} \binom{n-m}{n-m-k}$ where $0 \leq m < n$

$= \binom{n}{m} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{n-m-k}$ as $\binom{n}{m}$ is independent of K.

$= \binom{n}{m} \left((1-1)^{n-m} \right) = \binom{n}{m} \times 0 = 0$

d) $\sum_{k=0}^{m} \binom{n}{k} \binom{n}{m-k}$

$= \binom{n}{0} \binom{n}{m-0} + \binom{n}{1} \binom{n}{m-1} + \ldots + \binom{n}{m} \binom{n}{0}$

The expression means the selection of m elements out of two groups each consisting of n objects. This is as well can be taken as the selection of m out of 2 n elements.

$= \binom{2n}{m}$

Alternatively left hand side represents the coefficient of x^m in the expansion of $(1+x)^n (1+x)^n = (1+x)^{2n}$. By the question $m \leq n$.

Hence the result :

When $m = n$ $\binom{n}{m-k} = \binom{n}{n-k} = \binom{n}{k}$. Hence

$\sum_{k=0}^{m} \binom{n}{k} \binom{n}{m-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} =$

$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

TRANSFORMATION GEOMETRY

1. Congruence

The following congruence concept for five types of geometric figures are familier.

i) Congruence of segments

Two segments $\overline{AB}$ and $\overline{CD}$ are said to be congruent (written as $\overline{AB} \cong \overline{CD}$) if length of $\overline{AB}$ is equal to the length of $\overline{CD}$. Without any fuss we write them as; AB = CD

ii) Two angles ABC, DEF are said to be congruent if

$m \angle ABC = m \angle DEF$

iii) Two triangles $\triangle$ ABC and DEF are congruent if

a) $\exists$ 1-1 correspondences among vertices, i.e.,

$A \leftrightarrow D, \quad B \leftrightarrow E, \quad C \leftrightarrow F.$

b) pair of corresponding segments are congruent, i.e.,

$\overline{AB} \cong \overline{DE}, \quad \overline{BC} \cong \overline{EF}, \quad \overline{AC} \cong \overline{DF}.$

c) Pair of corresponding angles are congruent, i.e.

$\angle ABC \cong \angle DEF$
$\angle BCA \cong \angle EFD$
$\angle CAB \cong \angle EDF$

iv) Two circles are congruent if they have the same radius.

v) Two circular arcs are congruent if

a) the circles in which the arcs lie are congruent

b) the arcs have the same degree measure.

Euclid based all of his congruence proofs on the idea of congruence that "things which coincide with one another are equal to one another". This idea has been rephrased by some as "geometric figures can be moved without changing their size or shape".

The difficulty about this concept is that though the word 'figure' is regarded as a set of points, the terms "moved", "size" and "shape" do not have any meaningful status. We dispense with this difficulty by defining rigid motion or isometry.

Definition: Let M and N be sets of points and let

$$f: M \leftrightarrow N$$

be a one-to-one correspondence such that

$$d(f(P), f(Q)) = f(P)\,f(Q) = PQ = d(P,Q).$$

for points $P,Q \in M$, where $f(P)\ f(Q)$ denotes the distance $d(f(P), f(Q))$ in the set N.

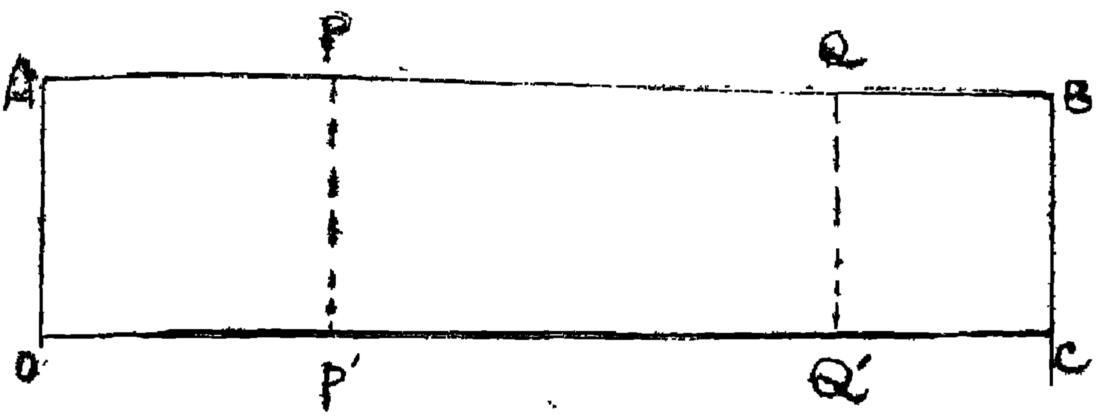
Then f is called a rigid motion or an isometry between M and N.

If there is an isometry between M and N, then we say that M and N are isometric and we write

$$M \approx N$$

Theorem 1. Opposite sides of a rectangle are isometric.

Proof:



Let f: $\overline{AB} \longleftrightarrow \overline{DC}$ (Fig.1)

be the vertical projection. Then for each point P $\overline{AB}$, $f(P) = P'$ is the foot of the perpendicular from P to DC. It is easy to verify that this establishes a 1-1 correspondence between $\overline{AB}$ and $\overline{DC}$. Besides this transformation has the special property that $d(P,Q) = PQ = d(\ f(P), F(Q)\ ) = P'Q'$ (Q.E.D.)

Problems:

1. Two segments of different lengths are never isometric.
2. An angle and a line are not isometric.
3. Two rays are always isometric.
4. Two circles of different radii are not isometric.
5. Let L and L' be two lines in the same plane, and let.

$$f: L \longleftrightarrow L'$$

be the vertical projection of L onto L' show that

$$L \parallel L' \Longleftrightarrow f \text{ is an isometry.}$$

6. Isometry is an equivalence relation.

The following theorem is more general than Theorem 1.

Theorem 2 If $\overline{AB} \cong \overline{CD}$, then $\exists$ an isometry $f: \overline{AB} \leftrightarrow \overline{CD}$ such that

$$f(A) = C, \quad f(B) = D.$$

Proof: Set up a co-ordinate system for $\overline{AB}$ and $\overline{CD}$ in such a way that co-ordinate of A and C are 0 and the co-ordinates of B and D are positive. Clearly the co-ordinate of B is the same as co-ordinate of D which is AB (= CD). Define f such that if $P \in \overline{AB}$ and has co-ordinate x, then $f(P) = P' \in \overline{CD}$ should have the co-ordinate x. Then if $Q \in \overline{AB}$ has co-ordinate y, then Q' has also the co-ordinate y. Hence

$$PQ = |x - y| = |f(P) - f(Q)| = P'Q' \quad \text{(Q.E.D.)}$$

Theorem 3: Given the correspondence

$$ABC \leftrightarrow DEF$$

between the vertices of two triangles. If

$$\triangle ABC \cong \triangle DEF,$$

then $\exists$ an isometry

$$f : \triangle ABC \leftrightarrow \triangle DEF$$

such that $f(A) = D$, $f(B) = E$, $f(C) = F$.

Proof: Let by Theorem 2,

$$f_1 : \overline{AB} \leftrightarrow \overline{DE}$$

be an isometry induced by the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$. Similarly let

$$f_2 : \overline{BC} \leftrightarrow \overline{EF}$$

$$f_3 : \overline{AC} \leftrightarrow \overline{DF}$$

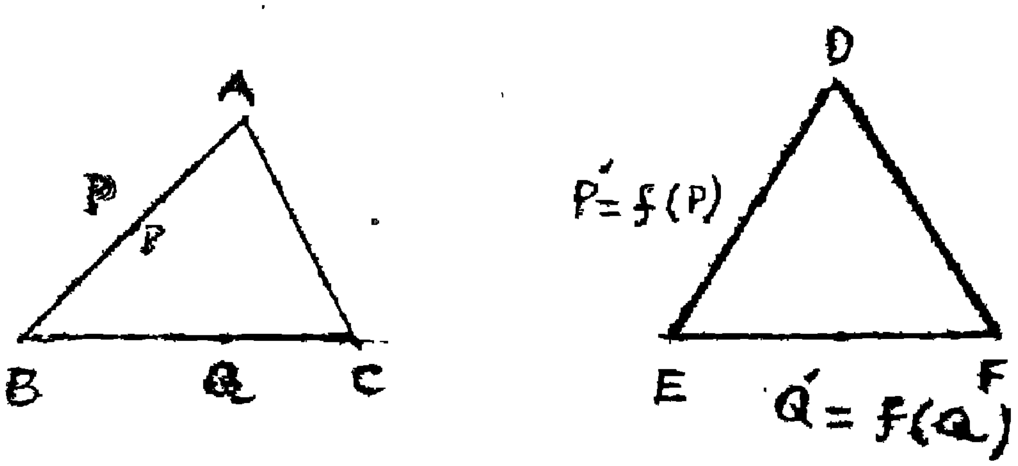
Let

$$f = \begin{cases} f_1 & \text{if } P \in \overline{AB} \\ f_2 & \text{if } P \in \overline{BC} \\ f_3 & \text{if } P \in \overline{CA} \end{cases}$$

If $P, Q \in \overline{AB}$ (say) then originally

$PQ = f(P)\, f(Q) = f_1(P)\, f_1(Q)$ ($\because$ f_1 is an isometry)

If $P \in \overline{AB}$ and $Q \in \overline{BC}$, then



(Fig.2)

$\overline{BP} \approx \overline{EP'}$ (as f_1 is an isometry)

$\overline{BQ} \approx \overline{EQ'}$ (as f_2 is an isometry)

Also

$\angle PBQ \approx \angle P'EQ'$ ($\because$ $\triangle ABC \approx \triangle DEF$)

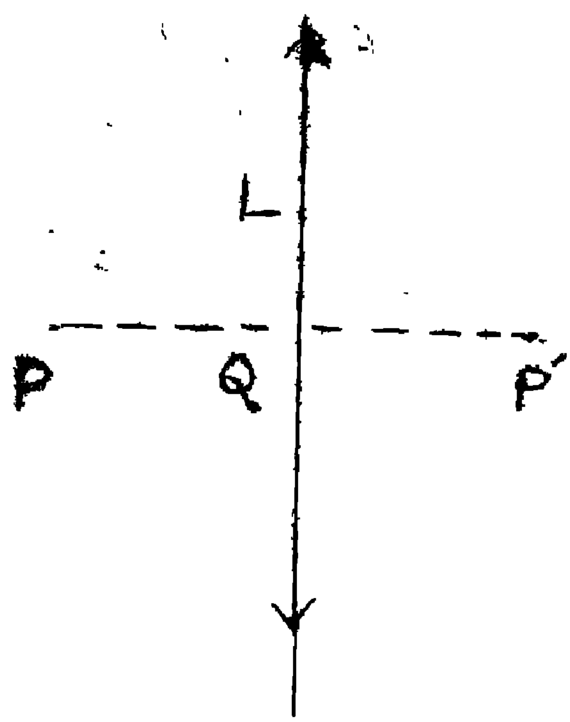
$\therefore$ By SAS, we have

$\triangle PBQ \approx \triangle P'EQ'$

So that

2. There are various types of isomorphisms that build up the transformation geometry. We shall discuss some common transformations.

a) Reflection: Let L be the line and let P be the point in the plane of the line, let $\overleftrightarrow{PQ} \perp L$ and let PQ = QP' (see the figure). P' is called the image of the point P above the line L. The map $M_L : P \rightarrow P'$ is called the reflection of P on the line L. L is called the axis of reflection.



(Fig.3)

Then reflection is a transformation that has the following properties:

i) To every line L, $\exists$ unique reflection M_L such that if P' is the image of the reflection of the object P, then P is the image when the object is P'. Thus

$M_L(P) = P'$ $\therefore$ $M_L(P') = P$

and P and P' lie on opposite sides of L.

ii) $M_L(L) = L$, that is, the line L is the fixed line of the mapping M_L. The line $\overleftrightarrow{PP'}$ is also a fixed line $\perp$ L.

iii) M_L leaves the distances and angles invarient, i.e. if

$M_L(\overline{PQ}) = \overline{P'Q'}$

then $PQ = P'Q'$

If $M_L(\angle A) = \angle A'$

then $m\angle A = m\angle A'$

iv) Given any two points P and Q in the plane, $\exists$ a reflection M_L such that $M_L(P) = Q$ and $M_L(Q)=P$

v) Given rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$, $\exists$ reflection M_L such that

$M_L(\overrightarrow{PA}) = \overrightarrow{PB}$, $M_L(\overrightarrow{PB}) = \overrightarrow{PA}$.

Here L is the bisection of the angle $\angle APB$.

b) <u>Composite reflections:</u>

If M_L and M_L' represents two reflections on the lines L and L', then the <u>composite reflection</u> $M_LM_{L'}$ is defined by

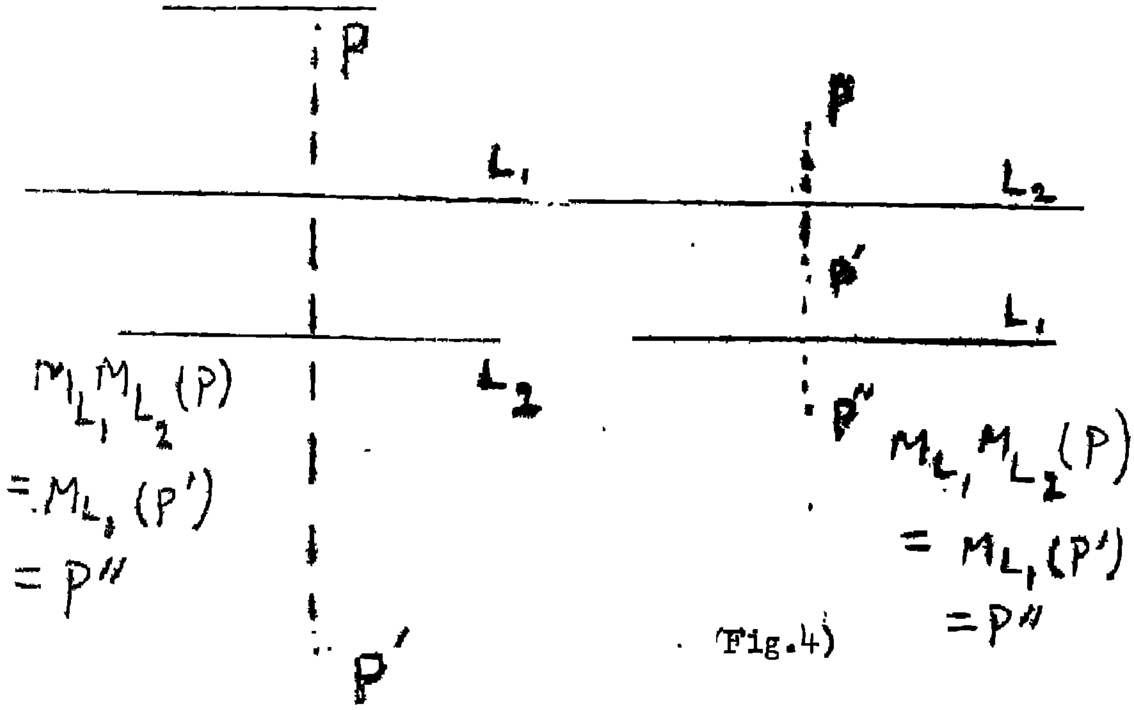
$$(M_LM_{L'})P = M_L(M_{L'}(P))$$

Note that

$$M_L^2(P) = (M_LM_L)(P) = M_L(P') = M_L(P)$$

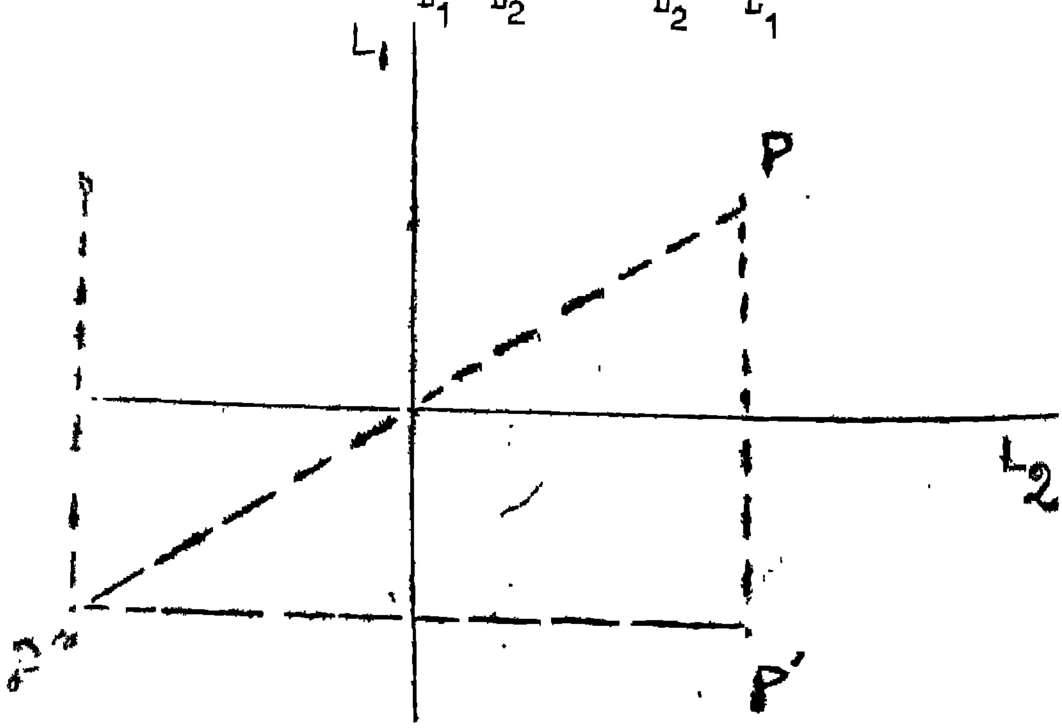
So that M_L^2 is the <u>identity transformation</u> in which all points of the plane continue to occupy the same position.

i) If $L_1 \parallel L_2$, then $M_{L_1} M_{L_2}$ represents a translation (see the figure 4)



(Fig.4)

ii) If $L_1 \perp L_2$, then it is easily computed that (see the figure 5) $M_{L_1} M_{L_2} = M_{L_2} M_{L_1}$



(Fig.5)

However, in general $M_{L_1} M_{L_2}$ is non-commutative.

(iii) Half turn (Rotation)

In the case $M_{L_1} M_{L_2} \neq M_{L_2} M_{L_1}$

then $M_{L_1} M_{L_2}$ is equivalent to a rotation about O

(where L_1 intersects L_2) (see Fig.5) which is also the middle point of the segment $\overline{PP''}$

This rotation is a half-turn about O and is usually denoted by H_0. Thus

$$H_0^2 = H_0 H_0 = M_2 M_{L_2} M_{L_2} M_{L_1} = M_{L_1} I M_{L_1}$$
$$= M_{L_1} M_{L_1} = M_{L_1}^2 = I \qquad (\text{as } M_{L_2}^2 = I$$

where I is the identify transformation.

(c) Linear Symmetry

The reflection M_L maps a figure onto itself, and so we say that L is the axis of symmetry of the figure. Since the figure is mapped onto itself by a half turn H_0, then we say that O is the centre of symmetry.

It is evident that the linear symmetry transforms a segment into a congruent segment and an angle into a congruent angle.

Thus reflections are transformations which leave distances and angles unchanged. Hence if a figure is mapped by a product of a finite number of reflections, then its image is congruent to the original. Thus finite product of reflections are congruences or isometries.

Problems:

1. Rectangle has two lines of symmetry; square has four lines of symmetry; circle has any line passing through the centre as a line of symmetry.
2. Obtain axis of symmetry of two given points.
3. Draw the axis of symmetry of an angle.
4. Given two intersecting circles, show that the line passing through their centres is the line of symmetry of points of their intersections.
5. How many circles can be drawn through two given points?

(d) Translation

Consider two successive reflections on two parallel axes L_1 and L_2 (see Fig.6) of the point P. Now

$$M_{L_1}(P) = P', \quad M_{L_2}(P') = P''$$

Thus the final image P'' is shown as in the figure 6. Note that

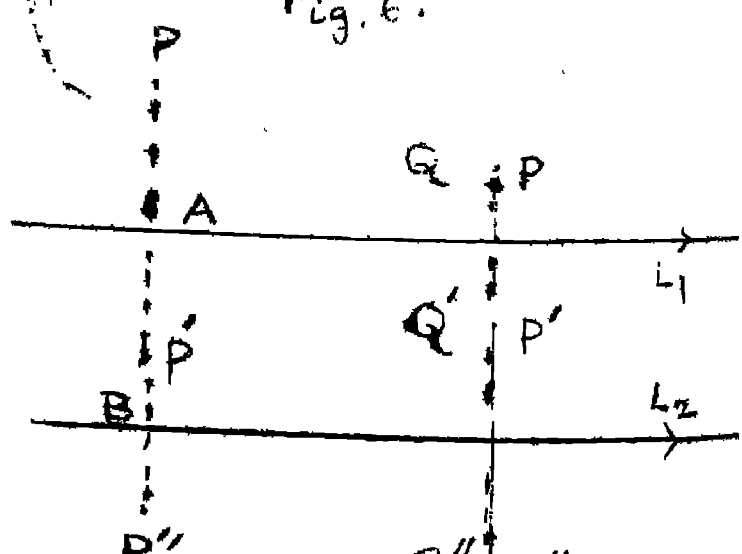
d = distance between L_1 and L_2

$= AB = AP' + P'B$

$= \frac{PP'}{2} + \frac{P'P''}{2}$

$= \frac{PP''}{2}$

Fig. 6.



Similar will be the case for any other point Q on the plane. Thus

$$d = \frac{PP''}{2} = \frac{QQ''}{2}$$

i.e. the composite reflection $M_{L_1} \cdot M_{L_2} = T$ where $L_1 \parallel L_2$

moves all points the same distance in the same direction. Such a map T is called a translation. Then a translation is defined as a the composite reflections on parallel lines.

Theorem: A translation is equivalent to two half-turns.

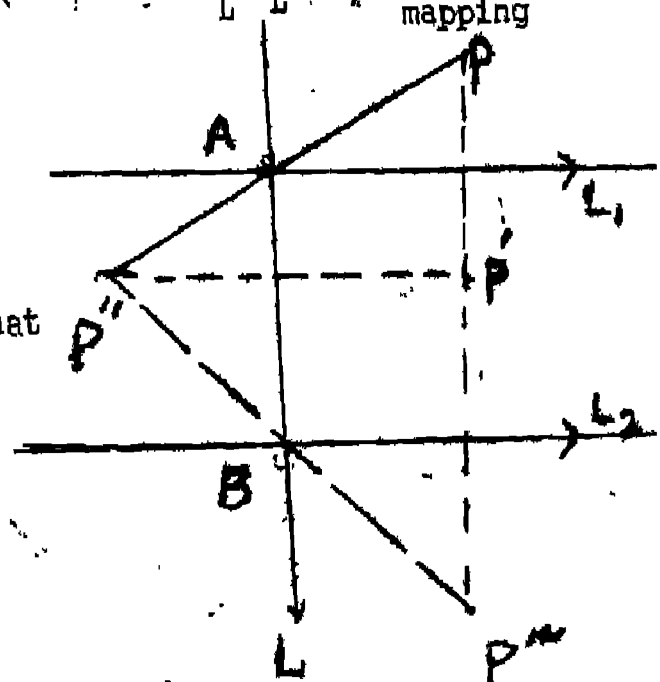
Proof: See $L_1 \parallel L_2$ and L be a transversal such that $L_1 \perp L$. Now the transelation.

$$T = M_{L_1} M_{L_2}$$

$$= M_{L_1} (M_L M_L) M_{L_2} \quad (\because M_L M_L = 1) \text{ Identity mapping}$$

$$= (M_{L_1} M_L) (M_L M_{L_2})$$

$$= H_A \cdot H_B$$



Verify from figure 7 that

$$T(P''') = M_{L_1} M_{L_2} (P''')$$

$$= M_{L_1} (P') = P$$

And

$$H_A \cdot H_B(P''') = H_A(P'') = P (Q.E.D.)$$

$$\text{Cor. } 2 H_A H_B = 2 \vec{V} = T$$

(Fig.7)

that is, the translation T is a vector $2\vec{V}$ where $\vec{V}$ is the vector from A to B with magnitude AB.

ROTATION

Rotation is a kind of transformation in which points in the plane are moved through a given angle θ about a fixed point 'O'.

If R_θ denotes the rotation map, then

$R_\theta(P) = P'$ (See fig.8)

It is easy to see that

$R_\theta \; R_\phi = R_{\theta+\phi}$

In particular

$(R_\theta)^2 = R_{2\theta}$

$R_\theta \cdot R_{-\theta} = R_0 = I$ (Identity)

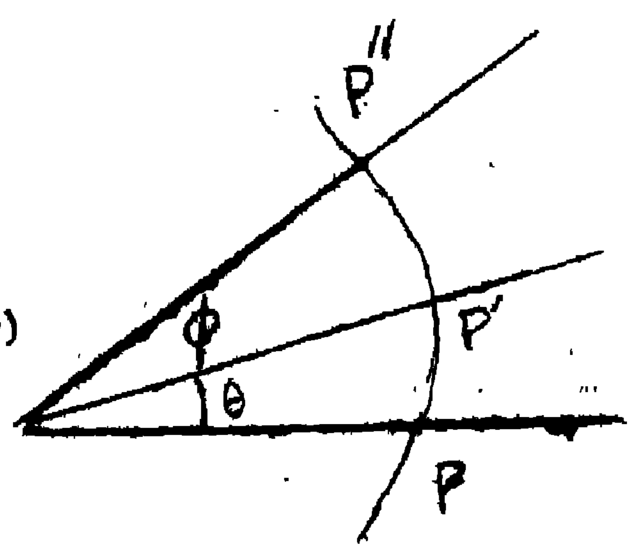


Fig. 8.

Verify from fig.8 that

$R_\theta(P) = P'$

$R_\phi(P') = P''$

$R_{\theta+\phi}(P) = P''$

$\therefore P'' = R_\phi(P') = R_\phi R_\theta(P)$

AXIOMATISATION OF MATHEMATICS

The axiomatic method in mathematics, which started with Euclid's Elements and was revived in the 19th Century has made an enormous progress since the beginning of 20th Century; almost all fields of mathematics and logic and some physics and other sciences, have since undergone an axiomatic analysis.

Cantor's theory of Ordinals and Cardinal numbers was the culmination of three decades of research on number "aggrega [illegible]" and he published his master works on ordinal and [illegible]nal numbers in 1895 and 1897. [illegible] In a series of papers initiated in 1874 Cantor developed an intuitively grounded general theory of sets treating particularly those sets having infinitely many members. Using this new theory of infinite, he proved some theorems which startled the mathematical community. Some of the theorems are:

(i) the set of all algebraic number is denumerable;

(ii) the set of all real numbers is not denumerable;

(iii) $\overline{A} < \overline{\overline{P(A)}}$

The notion of an infinite set as a complete and single entity was not universally accepted. Critics argued that logic is an extrapolation from experience that is necessarily finitistic. To extend the logic of the finite to the infinite entailed risks too grave to countenance. This prediction of logical disaster seemed vindicated when at the turn of the century, paradoxes or antimonies were discovered in the very foundation of the new discipline.

Dedekind stopped publication of his work and so also Ferge conceded that the foundation of his works has been destroyed by the Russell's paradox.

Foremost among the critics is Kronecker (who belonged to the Intuitionistic school) challenged Weirstrass and Cantor for the use of infinite sets. According to him, the theory of transfinite numbers is mysticism but not mathematics. Before Cantor, mathematicians like Abel, Gauss Cauchy and Weierstrass also used infinite sets; so also vaguely the 18th Century mathematicians Leibnitz, Newton, Bernoullii. But Cantor was first to systematically study the theory of sets (both finite and infinite). His vague and pseudo-definition of set is "a collection into a whole of definite, well-distinguished objects of our intuition or thought". Mathematicians did not work long with Cantor's ideas before a crisis developed. It was possible to formulate certain statements about sets which are contradictory; that is, both the statements and their denials are provable. One of the first and certainly the simplest of these contradictions was discovered by Bertrand Russell in 1900 by taking the collection of all sets $S \notin S$. Russell reasoned as follows:

Russell's Paradox:

Given a set S and an object x, the rules of logic dictates that either

$$x \in S \quad \text{or} \quad x \notin S.$$

In particular, take the object to be S, we have either

$$S \in S \quad \text{or} \quad S \notin S.$$

Let $R = \{ S : S \notin S \}$

Theorem

$$R \in R \Rightarrow R \notin R$$
$$R \notin R \Rightarrow R \in R$$

The appearance of such paradoxes as Russellls, precipitated a search for a rigorous foundation of set theory which would be devoid of contradiction. Cantor's theory of sets needed drastic revision. What was needed was a precisely stated system of axioms saying enough about the behaviour of sets to capture the intuitive meaning of 'set' and yet, hopefully to so delimit this concept as to avoid paradox.

The earliest attempt to axiomatize Cantor's naivė set theory was that of G.Ferge (1893-1903). He included a so-called axiom of abstraction, which asserts the existence, for any given property P of a set whose members are precisely those objects having the property. If we take the property P to be $S \notin S$ then Russell paradox immediately results. Of course, Russel's paradox is an immediate consequence of that axiom and as a result Ferge abandoned the publication of his work.

According to Cantor and Ferge, a set was thought of being defined by a property. But as we have seen, it is not enough to specify a set to pronounce some magic words like $S \notin S$.

Nevertheless set theory gained sufficient support to survive the crisis of the paradoxes. In 1908, speaking at the international congress at Rome, the great Henri Poincare urged that a remedy be sought. As a reward he promised the 'joy of the physician called to treat at beautiful pathologic case". By that time Zermelo and Russell were already at work seeking fundamental principles on which a consistent theory could be built.

From this, one might assume that the sole purpose for axiomatizing is to avoid the paradoxes. There are however reasons to believe that axiomatic set theory would have evolved even in the absence of paradoxes. Certainly the works of Dedekind and of Ferge in the foundations of arithmetic was not motivated by fear of paradoxes but rather by a desire to see what fundamental principles are required.

Axiomatization of Set Theory is appropriate for three reasons:

(i) The antimonies of Set Theory which appeared at the turn of the 20th Century showed that the quasi-constructive procedure of Cantor's set theory has to be restricted in some way; over-comprehensive sets had to be ruled out.

(ii) The fact that all other branches of mathematics can be incorporated in set theory, leads to the idea of setting up of a comprehensive axiom system of set theory in which the axiomatic disciplines of other branches can be embedded.

(iii) By analysing the mathematical arguments logicians became convinced that the notion of 'set' is the most fundamental concept of mathematics.

In developing a theory of set we have two alternatives. Either we must abandon the idea that our theory is to encompass arbitary collections in the sense of Cantor, or we must distinguish between at least 2 types of collections; arbitary collections that we call classes and certain special collections that we call sets. Classes or ~~abc~~ arbitary collections, are however so useful and our intuitive feelings about classes are so strong that we dare not abandon them. A satisfactory theory of sets must provide a means of speaking safely about classes. There are several ways of developing such a theory.

(1) R-W System

Russell and Whitehead in their principia mathematics (1910) resolved the difficulties with a theory of types. They established a hierarchy of types of collections. A collection x can be a member of a collection y only if y is one level higher in the hierarchy than x. In this system there are variables for each type level in the hierarchy and hence there are infinitely many primitive notions.

(2) Z-F System (model)

Zermelo (1908) -Fraenkel (1922) set theory accepts two primitive notions: set and membership. Class is introduced as a defined term in this scheme. In the formal language, we have only 'set' variables and a binary predicate symbol '$\in$'.

(3) G-B System (model)

In Godel (1940) - Bernays (1937-1954) system of set theory, there are three primitive notions: set, class, membership. In this system, the paradoxes like that of Russell are avoided by recognizing two types of classes: sets and proper classes. Sets are classes that are permitted to be members of other classes. Proper classes have sets as elements but are not themselves permitted to be elements of other classes. A Z-F quantification is permitted only on set variables while G-B quantification is permitted on both set and class variables, there are theorems in G-B that are not theorems in Z-F. It can however be proved that G-B is a conservative extension of Z-F in the sense that every well formed formula (wff) of Z-F is provable in Z-F, if and only if it is provable in G-B.

EVOLUTION OF THE AXIOMATIC METHOD

Euclid's Elements (written about 300 B.C.) is the first written evidence of axiomatic method in mathematics. It contains a groupd of 'axioms' and 'postulates'. An axiom was used to be thought of as self-evident truth and a postulate was thought of as a simple geometrical fact whose validity may be assumed. From these, Euclid deduced 465 propositions in a logical chain.

Logical deductions were common among the other scholars of the period. They include Aristotle (384-321 B.C. L.C.) Plato, Pythagoreous.

Axiomatic method found in Euclid perhaps was an attempt to overcome the difficulties arising out of the discovery of irrationals and the paradoxes of Zeno.

Archimedes (287-212 B.C.) used the axiomatic method of Euclid in his two books on theoretical mechanics.

Newton's famous Principia first published in 1686 is organised as a deductive system. The treatment of analytic mechanics published by Lagrange in 1788 is a masterpiece of logical perfection.

Axioms and postulates were thought of as having the character of "logical necessity". Surely if a statement is a logical necessity, the assumption of its invalidity should lead to contradiction. But Euclid's fifth postulate wasproved independent of all other axioms, that is, this postulate cannot be demonstrated as a logical consequence of other axioms and postulates in the Euclidean system. By a suitable replacement of the fifth postulate, we have now non-euclidean geometries.

Hilbert published in 1899 the finest work of his times on the foundations of geometry where he used "axioms" and "undefined" terms.

SET THEORY

Z-F System

Within a certain non-empty domain of objects we take as the only primitive relation of our axiomatic system Z-F, the membership relation $\in$. If x and y denote any objects in the domain, then either $x \in y$ or $x \notin y$ will hold true.

Now given the propositional calculus and predicate calculus, we now examine the concept of EQUALITY explicitely.

The following attitudes are possible.

a) Equality in its logical meaning as identity. Zermelo adopts this attitude by calling x and y equal if they denote the same thing (object). When objects are sets, he in addition rests on an axiom of extensionality which states that a set is determined by its elements.

b) Equality as a second primitive relation within Z-F. Then usual properties of equality has to be guaranteed axiomatically. The usual properties of equality are:-

i) Substitutivity with regard to $\in$

ii) Extensionality.

c) Equality as a mathematically defined relation. We may define $x = y$.

either by 'if every set that contain x contains also y and vice-versa".

or by 'if x and y contain the same elements".

The second way is possible if every object is a set (including the null-set). In the former case, extensionality must be postulated axiomatically; in the latter,

an axiom has to guarantee the former property.

We adopt the method (c) which is superior to (b) in so far as a single primitive only occurs in the system and to (a) since the system is constructed upon a weaker basic discipline.

We now admit a set ''null-set'' into the domain of Z-F system.

Axiom of Null-Set:

$$\exists x : \forall y \; (\nexists y \in x).$$

We denote this x by ϕ

Now there are four possibilities about the objects or individuals for the domain. The domain contains:

i) ϕ and also other individuals

ii) other individuals but no null-set

iii) one null-set and no other individuals

iv) no null-set and no individuals.

Case (iv) is impractical. Case (i) was proposed by Zermalo, case (ii), by Quine (1936) and case (iii) was first proposed by Fraenkel (1921/22) and later accepted by Von Newmann, Bernays and others. We adopt here the case (iii).

The following is the sequence of steps in the development of Z-F system:

1. All objects of Z-F are sets.
2. The empty -set ϕ is one of the objects of Z-F system. It could be the only object of Z-F system. This is an ad-hoc axiom; it will be superseded by more promising axioms later.

3. The only primitive relation is the dyadic relation $\in$ of membership whose arguments are sets; that is, if x and y denote any objects in the domain, then the statement $x \in y$ shall either hold true or its negation $\neg(x \in y)$ (written as $x \notin y$) shall hold true.

Def (Subset) : Let s and t be sets.
If $\forall x\,(x \in s \Rightarrow x \in t)$, then s is called a subset of t and we write this as $s \subset t$.

This definition does not allow us to construct a new set out of old ones by collecting some of its elements, in contrast to cantor's comprehensive method of construction of sets. Only when two sets are given, we may state whether one is a subset or not of the other.

Def. (Equality):

$$s = t \quad \text{iff} \quad \forall x (x \in s \Leftrightarrow x \in t).$$

That is to say, sets are equal if contained in the same objects (sets).

Axiom 1 (Axiom of Extensionality)

$$s \subseteq t \wedge t \subseteq s \Rightarrow s = t$$

That is

$$\forall x\,(x \in s \Leftrightarrow x \in t) \Rightarrow s = t.$$

That is, two sets s and t are equal if they have the same elements. Thus a set is determined by its extension (by its elements). We denote a set with elements a, b,c........ by $\{a, b, c\ldots\ldots\ldots\ldots\ldots\}$.

The following results follow from the axiom of extension.

Theorem 1

i) The set is unaffected by the order of elements. That is, if B is the set obtained by listing the elements in different order given in A, then

$$A = B.$$

ii) The set is unaffected by the repetition of elements. Thus

$$\{a, a\} = \{a\}$$

iii) Empty set

ϕ is unique

iv) a $a = a$ (reflexive)

v) $a = b \Rightarrow b = a$ (symmetry)

vi) $a = b \wedge b = c \Rightarrow a = c$ (transitive).

Theorem 2 (Theorem of Substitution) :

$$x \in s \wedge s = t \Rightarrow x \in t.$$

A. Robinson (1939).

It is valuable to understand that the axiom of extension is not just a logically necessary property of equality but a non-trivial statement about belonging. To understand this we take the following example.

We write

$$x \in s$$

If x is an ancestor of s.

The analogue of the axiom of extension would say here that two human beings are equal if and only if they have the same ancestor.

Clearly 'if' part is false.

We observe that belonging ($\in$) and inclusion ($\subseteq$) are conceptually very different things.

i) Inclusion is reflexive always but it is not at all clear whether belonging is ever reflexive.

ii) Inclusion is transitive, whereas belonging is not, $x \in y$, $y \in t$ does not mean that $x \in t$.

Axiom 2 (axiom of pairing)

For any two sets a and b, the pair $\{a, b\}$ is or $\{b, a\}$ exists as sets. Thus given two sets, there is a third set that they both belong to.

And on account of axiom of extensionality,

$$\{a, b\} = \{b, a\}$$

and therefore we are entitled to use a definite pair.

If a is an object in our system, they by axiom of pairing (and axiom of extension).

$\{a\} = \{a, a\}$ is a set, So also is $\{\{a\}\}$, $\{\{\{a\}\}\}$ are sets; by repeated applications of axiom of pairing we obtain

$$\{a, \{a\}\},$$

$$\{a, \{a\}, \{a, \{a\}\}\}$$ as sets.

This axiom is a limited instrument for construction of new sets from old ones.

Axiom-3 (Axiom of Sum-set) (Axiom of Union)

Given any set $x \exists y$

$\forall x \exists y : \forall z\ (z \in y \Leftrightarrow t \in x \mid z \in t)$

For any set x which contains at least two elements, there exists a set whose elements are the elements of the elements of x. Suppose that

$$x = \{a, b\}$$

Then $\exists$ a set $y : \forall z\ (z \in y \Leftrightarrow \exists t \mid z \in t$ and $t \in x)$

Here $t = a$ or b

Hence $\exists$ a set $y : \forall z\ (z \in y \Leftrightarrow z \in a \vee z \in b)$

If $x = \{a, b, c\}$

then $\bigcup x = a \cup b \cup c$.

Theorem-1 Union of two sets is a set.

Proof Suppose that a and b are sets. Then by axiom 2, $\{a, b\}$ is a set and then by axiom 3 $a \cup b$ is a set.

Theorem-2 Let a, b, c be sets. Then

$$a \cup (b \cup c) = (a \cup b) \cup c = a \cup b \cup c.$$

Proof By Theorem 1 $a \cup b$ and c are sets and hence by Theorem 1 again $(a \cup b) \cup c$ is a set.

Axiom 2 and 3 are also limited instruments for set construction in so far as they produce at best "denumerable" number of new sets.

·4

Axiom 4 (Axiom of Power set)

For any set s, there exists the set whose elements are all subsets of s.

This set is called the power-set of s and is denoted by P(s).

Given a set, we have not yet any instrument to construct subsets and therefore at this stage the axiom of powerset is also limited; but this becomes a powerful method of set construction after we have axiom 5.

Remark: If we know that a is a set, then axiom 1 and 2 gives that $\{a\}$ is a set.

Conversely also if $\{a\}$ is a set, then a is a set (by universe of discourse). Similarly for 2 sets a and b. But if we have three sets a, b, c then is it that $\{a, b, c\}$ is a set?

Axiom of pairing does not ensure it. For this shall we fall back upon Axiom of Specification?

If a, b, c are sets, then $a \cup b$, c are sets and so $s = a \cup b \cup c$ is a set. By axiom 4, P(s) is a power set. Since $a \subset s$, $b \subset s$, $c \subset s$, it follows that $a, b, c \in P(s)$.

Now take the predicate P to be

$$\text{“} (x = a) \vee (x = b) \vee (x = c). \text{”}$$

Then the set

$$\alpha = \{x \in P(s) : P(x)\}$$

exists and the set is

$$\alpha = \{a, b, c\} \subset P(s).$$

Axiom 5 (Axiom of Subsets or specification)

'For any set s and any predicate P which is meaningful ('definite') for all elements of s, there exists the set y that contains just those elements x of s which satisfy the predicate P.'

Axiom 6 (Axiom of Choice)

'For every set t for which $\phi \notin$ t, there exists the set whose elements are the sets which contain a single element from each element of t'.

Axiom 7 (Axiom of Infinity) Zermelo (1908)

$\exists$ at least one set w such that

i) $\phi \in w$;

ii) $x \in w \Rightarrow \{x\} \in w$.

REFERENCE BOOKS

1. Set Theory and the Continum Hypotehsis:
   Paul J.Cohen, W.A.Benjamin, Inc. (1966).

2. Introduction to Axiomatic Set Theory:
   G. Takeuli & W.M.Zaring, Springer- Verlag (1971).

3. Axiomatic Theory of Sets and Classes;
   Murray Eisenberg (1970)

4. Axiomatic set Theory, P.Suppes Van Nostrand,
   Princeton (1960)

5. Introduction to the Foundations of Mathematics:
   R.L.Wilder (1965).

ROLLE's THEOREM AND MEAN VALUE THEOREMS AND THEIR GEOMETRICAL INTERPRETATIONS

In these write upswe shall discuss some important and useful theorems in differential calculus. Before we do this we need the following concepts.

Open interval: Let a and b be two real numbers such that $a < b$. Then the set $\{x \mid a < x < b\}$ constitutes an open interval. We also write this as (a,b). Note that $a \notin (a,b)$ and $b \notin (a,b)$ but any x such that $a < x < b$ is an element of (a, b).

Closed interval: Let a and b be two real numbers such that $a < b$. Then the set $\{x \mid a \leq x \leq b\}$ is a closed interval and is denoted as $[a, b]$. Here $a \in [a,b]$, $b \in [a,b]$ and any x such that $a < x < b$ is also an element of $[a, b]$.

Semiclosed intervals: The sets $a \leq x \quad b$ and $a < x \leq b$ are semiclosed intervals denoted respectively as $[a, b)$ and $(a, b]$. We may call them to be semi open intervals.

The points a and b are called the end points. In a closed interval the end points are included while in an open interval these are excluded.

Bounded function: A function f defined in an interval $[a, b]$ is said to be bounded if there exists numbers A and B such that

$$A \leq f(x) \leq B \text{ for any } x \in [a, b].$$

The numbers A and B are referred to as the lower and upper bound of the function in $[a, b]$.

Continuity and boundedness: We see intuitively that a function f defined and continuous in a closed interval $[a, b]$ is bounded. We do not prove this proposition.

But try to realise it. As f is continuous in the closed interval $[a, b]$ at every point of it the function assumes finite values. The greatest of these finite values is our B and the least is our A which are the bounds of f.

Now let f be continuous in $[a, b]$. Suppose $f(a) < f(b)$. If C is a number such that $f(a) < C < f(b)$ then does f assume the value C for some x in $[a,b]$? The answer is in affirmative and in fact there is a number c (a,b) such that $f(c) = C$.

To see this let us consider the function

$\phi(x) = f(x) - C$. Then

$\phi(a) = f(a) - C < 0, \phi(b) = f(b) - C > 0$

Now ϕ is a function which is continuous in $[a,b]$ and has values of apposite signs at a and b. Then the graph of the function i.e. $y = \phi(x)$ must intersect the x-axis somewherebetween $x=a$ and $x=b$. This implies that $\phi(x) = 0$ for some $x=c$ lying in between a and b, i.e. $\phi(c) = 0$ or

$f(c) - C = 0$ or $f(c) = C$ where $c \in (a,b)$.

Note that we have not really proved it but have tried to explain. However, the formal proof though somewhat similar to our explanation is a bit more delicate and difficult.

<u>Theorem</u> (Rolle's theorem)

<u>Statement</u>:- Let f be a function defined in the closed interval $[a,b]$ and satisfy the following :

1. f is differentiable in the open interval (a,b)
2. f is continuous in the closed interval $[a,b]$
3. $f(a) = 0 = f(b)$

Then there is at least one point $c \in (a,b)$ such that $f'(c) = 0$.

<u>Proof</u>: If the function f is constant, then $\frac{df}{dx} = 0$ for all $x \in [a,b]$ and so the theorem is trivially true. Now suppose that f is not constant.

As $f(a) = f(b) = 0$, the function assumes non-zero values which may be positive or negative. Further as f is continuous in $[a,b]$ and is as such bounded. Let M be the maximum of the function in $[a, b]$ and $M \neq 0$. Further let c be the point in (a,b) where $f(c) = M$.

Now let h be a small positive number. Hence

$$f(c \pm h) \leqslant f(c)$$

$$f(c + h) - f(c) \leqslant 0 \text{ and } f(c-h) - f(c) \leqslant 0$$

$$\frac{f(c+h)-f(c)}{h} \leqslant 0 \text{ and } \frac{f(c-h)-f(c)}{-h} \leqslant 0$$

But as h is arbitrary we can make it as small as we desire. Making $h \to 0$ as f is differentiable in (a,b) we have

$$f'(c) \leqslant 0 \text{ and } f'(c) \geqslant 0$$

$$\therefore f'(c) = 0$$

and this is what we required.

<u>Note</u>: If $f(a) = f(b) = k \neq 0$, then also Rolle's theorem is true. In this case we can consider the function

$$g(x) = f(x) - k$$

which is continuous in $[a,b]$, differentiable in (a,b) and $g(a) = g(b) = 0$. So there will be a point c in (a,b) such that

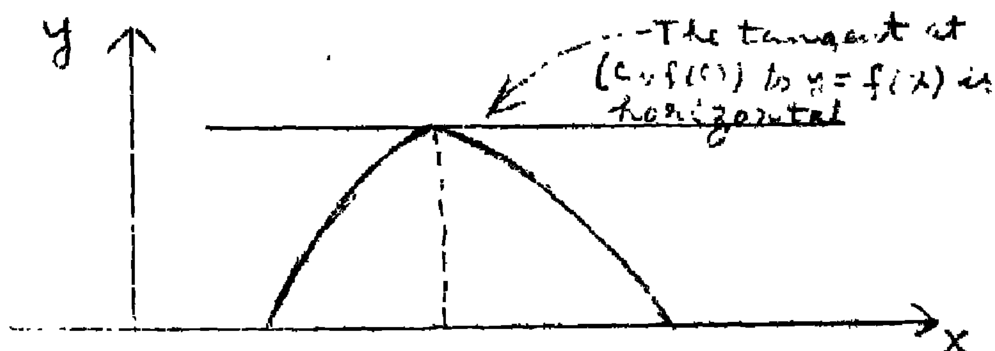
$$g'(c) = 0 \quad f'(c) = 0 \text{ as}$$

$$g'(x) = f'(x)$$

Geometrical meaning of Rolle's Theorem :

This theorem has simple geometrical meaning. We know that $f'(x)$ at a point x means $\tan\theta$ where θ is the angle that the tangent to the curve $y = f(x)$ makes with the positive side of the x-axis in the anti cyclic sense.

When $f'(x) = 0$ we mean $\tan\theta = 0 \Rightarrow \theta = 0$ i.e. the tangent to $y = f(x)$ at (x, y) is parallel to the x-axis or in other words horizontal.



Let us draw the figure obeying all conditions of the Rolle's theorem. Then we note that for some point $x = c$. We will see that the tangent at $(c, f(c))$ to $y = f(x)$ will be horizontal.

Example : Consider the function $f(x) = x^2 - 5x + 6$. Here f is continuous in $[2, 3]$ and differentiable in $(2, 3)$. In fact f being a polynomial continuous and differentiable throughout $\mathbb{R}$. Further $f(x) = x^2 - 5x - 6 = (x-2)(x-3)$. So

$$f(x) = 0 \text{ when } x = 2 \text{ and } 3 \text{ i.e.}$$

$$f(2) = f(3) = 0.$$

Therefore Rolle's theorem is applicable so far as f is concerned. Hence for some x in $(2, 3)$ we must have

$$f'(x) = 0$$

but $f'(x) = \frac{d}{dx}(x^2 - 5x - 6) = 2x - 5$ and

$2x - 5 = 0 \Rightarrow x = \frac{5}{2} \in (2, 3)$.

Example: Let $f(x) = \sin x$. Hemc f is continuous in $[(0, \pi)]$ and differentiable in $(0, \pi)$. Further $\sin 0 = \sin \pi = 0$

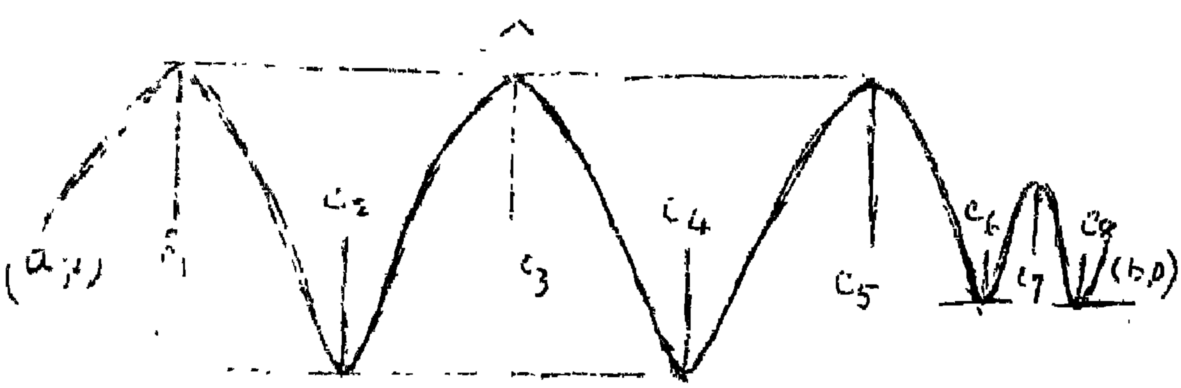
Hence all the conditions of Rolle's theorem are satisfied by the function $f(x) = \sin x$. Therefore we have

$\frac{d}{dx} \sin x = 0$ for some x in $(0, \pi)$

But $\frac{d}{dx} \sin x = \cos x$;

and $\cos x = 0$ at $x = \pi/2$ which is in $(0, \pi)$

Remark: Note that if all the conditions of Rolle's theorem are satisfied by the function f in an interval $[a,b]$ then we may have more than one point in (a,b) where $f'(x) = 0$, In the statement of the theorem we insist that there is at least one point where $f'(x) = 0$. This idea is illustrated through the following diagram.



Here, we have $f'(x) = 0$ at $x = c_1, c_2, c_3, c_4, c_5, c_6$ c_7 which are all inside (a,b).

Eercise:

In each of the following cases say whether Rolle's theorem is true. In case it is true find a suitable c where $f'(c)=0$.

1. x $f(x) = x-2, x \in [(0,1)]$
2. $f(x) = x, x \in [(-1,1)]$
3. $f(x) = 3x-x^3, x \in [0, \sqrt{3})]$
4. $f(x) = x^2+2x-3, x \in [(-3,1)]$

Rolle's theorem has many important applications. As an application of Rolle's theorem we now prove the mean value theorems. First we prove the Lagrange's mean value theorem. This theorem is also called as 'the law of the mean' or 'the formula of finite increments' (formule des accroissements finis).

Lagrange's mean value theorem:

Let the function f defined in $[a,b]$ be

1. continuous in $[a,b]$
2. differentiable in (a,b).

Then there is atleast one point c in the open interval (a,b) such that $f(b)-f(a) = (b-a)f'(c)$.

Note: Here f satisfies all conditions of Rolle's theorem except $f(a) = f(b)$.

Proof: Let $g(x) = f(x)-Ax$ where the constant A is such that $g(a) = g(b)$.

Further the function g is continuous in $[a,b]$ and differential in (a,b) as f satisfies these conditions by hypothesis. Now the constant A is given by

$$f(a)-Aa = f(b)-Ab$$

$$\text{i.e.} \quad A = \frac{f(b)-f(a)}{b-a} \qquad \ldots\ldots\ldots\ldots(1)$$

So all conditions of Roller's theoreum are satisfied by $g(x)$

But from Rolle's theorem there is atleast one c in (a,b) such that $g'(c)=0$.

i.e. $f'(c)-A = 0$

i.e. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ from (1)

i.e. $f'(c)(b-a) = f(b)-f(a)$.

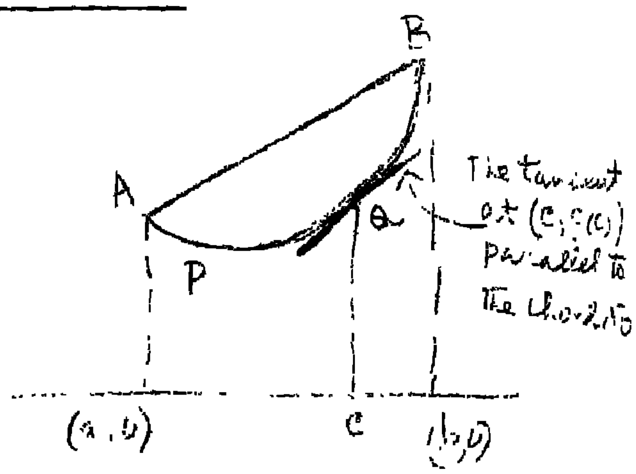
Remark: The point c may be mentioned as

$$c = a+\theta h.$$

Where $h = b-a$ and $0 < \theta < 1$ i.e. θ is a positive proper fraction. Hence the mean value theorem is

$hf'(a+\theta h) = f(b)-f(a)$, $h = b-a$ and $0 < \theta < 1$.

Geometrical meaning of the theorem:



The geometrical meaning of the mean value theorem is also simple. Let APQB be the graph of $y = f(x)$ in $[a,b]$. Let the chord AB joining the points $A(a,f(a))$ and $B(b,f(b))$ make an angle α with the x-axis. Then

$$\tan \alpha = \frac{f(b)-f(a)}{b-a}$$

By the mean value theorem we have

$$\tan \alpha = f'(c).$$

i.e. for some c in (a,b), the tangent to the curve $y = f(x)$ at $(c,f(c))$ is parallel to the chord AB or in other words the chord AB and the tangent at Q $(c,f(c))$ have same slope.

Example: Verify the mean value theorem for the function $f(x) = 3+4x-x^2$, $a=1$ and $b=4$

Here f is continuous in $[1,4]$ and differentiable in $(1,4)$. So there is a point c in $(1,4)$ such that

$$f'(c) = \frac{f(4)-f(1)}{4-1}$$

We shall now determine this c.

$f'(x) = 4-2x$. So $f'(c) = 4-2c$;

$f(4) = 3+4\times4 - 4^2 = 3$.

$f(1) = 3+4\times1-1^2 = 6$

$\therefore\quad 4-2c = \dfrac{3-6}{4-1} = \dfrac{-3}{3} = -1$

i.e. $2c = 5$ or $c = \dfrac{5}{2} = 2.5$

Hence for $x = 2.5$ in $(1,4)$ the tangent to the given curve $y = f(x)$ is parallel to the chord joining the points $(1,6)$ and $(4,3)$.

We can, using the mean value theorem, prove the following:

If the derivative $f'(x)$ is identically zero on the open interval (a,b) then $f(x)$ is constant on the interval.

Proof: Let x_1 and x_2 be two real numbers such that

$$a < x_1 < x_2 < b.$$

The function $f(x)$ is continuous in $[x_1,x_2]$ and differentiable in (x_1,x_2). So by Lagrange's mean value theorem we have

$f(x_2)-f(x_1) = (x_2-x_1)f'(c)$

for some c in (x_1, x_2).

But as $f'(x) = 0$ on (a,b) we have

$f(x_2) - f(x_1) = (x_2-x_1) f'(c) = 0$

i.e. $f(x_2) = f(x_1)$

So the function $f(x)$ has the same value at any two points x_1 and x_2 in (a,b) and consequently it is constant.

Exercise:

For each of the following problems find a value $x = c$ such that $f(b)-f(a) = (b-a)f'(c)$. If no such c exists in (a,b) state that condition of the mean value theorem is not satisfied.

Exercise:

For each of the following problems find a value $x = c$ such that $f(b)-f(a) = (b-a)f'(c)$. If no such c exists in (a,b) state what condition of the mean value theorem is not satisfied.

1. $f(x) = x^2$, $a = 1$, $b=4$
2. $f(x) = \frac{1}{x}$, $a = 1$, $b=5$,
3. $f(x) = x$ $a = -1$, $b=1$.

Cauchy's mean value theorem:

Let the two functions $f(x)$ and $g(x)$ be each continuous in $[a,b]$ and differentiable in (a,b) and $g'(x) \neq 0$ in (a,b), Then there is a point c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Let us define the function as

$$\psi(x) = f(x)-Ag(x) \qquad \ldots\ldots\ldots(1)$$

Where A is a constant such that

$$\psi(b) = \psi(a) \qquad \ldots\ldots\ldots\ldots(2)$$

Then (2) implies that

$$f(b)-Ag(b) = f(a)-Ag(a)$$

$$\text{i.e. } A = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad \ldots\ldots\ldots\ldots(3)$$

Now as $g'(x) \neq 0$ in (a,b) we have

$$g(b) \neq g(a)$$

For if $g(b) = g(a)$ then the function g will satisfy all the conditions of Rolle's theorem and for some $x=d$, in (a,b) $g'(d) = 0$. But as $g'(x) \neq 0$ in (a,b)we can not have $g(b) = g(a)$

Now as both f and g are continuous in $[a,b]$ differ ntiable in(a,b) , the function $\psi(x)$ is continuous in

$[a,b]$ and differentiable in (a,b). ψ satisfies the third condition (2) of Rolle's theorem if A is as given in (3). So the function ψ satisfies all the conditions of Rolle's theorem and hence there exists at least one point c in (a,b) such that

$$\psi'(c) = 0$$

i.e. $f'(c)-Ag'(c)=0$

i.e. $A = \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

Note: The Lagrange's mean value theorem is a particular case ofCauchy's mean value theorem. By getting $g(x)=x$ in Cauchy's mean value theorem we obtain the Lagrange's mean value theorem.

Example: For the pair of functions $f(x) = \sin x$ and $g(x)=x^2$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we can not apply Cauchy's mean value theorem. Even though the conditions of continuity and differentiability are satisfied we see that the condition $g'(x) \neq 0$ in $(-\pi/2, \pi/2)$ is not satisfied. For $g'(x) = \frac{d}{dx}(x^2) = 2x$ and $2x = 0$ at $x=0$ which is in $(-\pi/2, \pi/2)$.

On the contrary if we choose $g(x)=x$ we can apply Cauchy's mean value theorem and have

$$\frac{\sin(\pi/2) - \sin(-\frac{\pi}{2})}{\pi/2 - (-\pi/2)} = \frac{\operatorname{Cos} c}{1}$$

$$\frac{2}{\pi} = \operatorname{Cos} c$$

and $c = \operatorname{Cos}^{-1}\left(\frac{2}{\pi}\right)$

Increasing and decreasing functions:

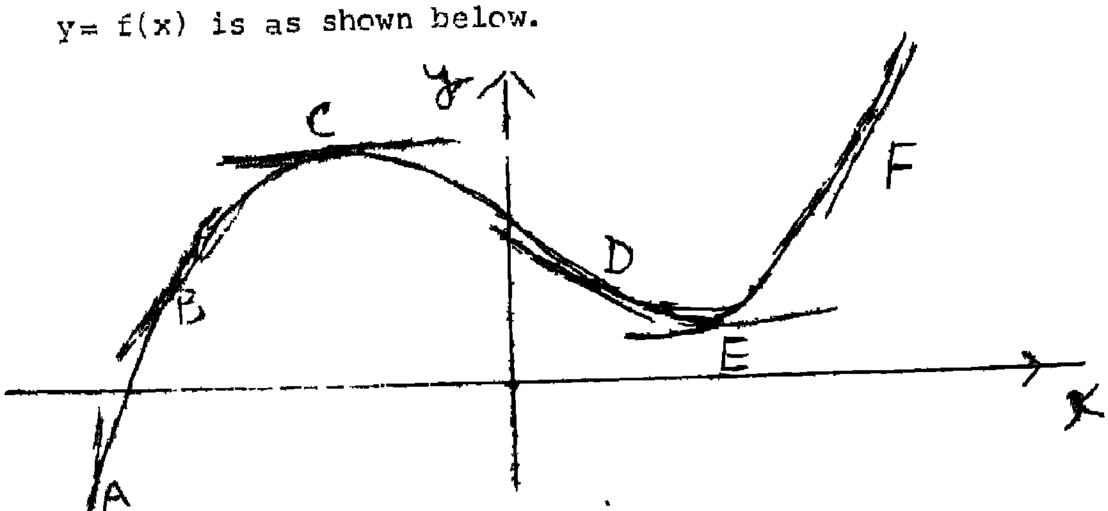
Definition: A function f is said to be increasing on an interval (a,b) if for any pair of points x_1 and x_2 in (a,b)

we have $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ and the function f is said to be decreasing in (a,b) if for any pair of points x_1 and x_2 in (a,b) we have $f(x_1) < f(x_2)$ when ever $x_1 > x_2$.

Let our function be differentiable. Then we shall see that increasing or decreasing character of the function (monotonicity of the function) in (a,b) can be determined from the sign of the derivative of the function f.

Before formulating the ~~xx~~ principle for monotonicity of the function f let us consider the following situation which would help us in stating the desired principle.

Suppose we consider the function f whose graph y= f(x) is as shown below.



If we think that the graph is traced as a point moves from left to right we note that the point would sometimes rise and sometimes fall and consequently we have the portion ABC rising the portion CDE falling and the portion EFK rising and so on. Let x- coordinates of A,B,C,D,E,F,K be a_1, a_2, a_3, a_4, a_5, a_6 and a_7 respectively. Then in (a_1, a_3) the function is increasing in (a_3, a_5) it is decreasing while in(a_5, a_7) it is increasing. Drawing tangents at A,B, C,D,E,F we note the following. Tangents

at A, B make acute angle with x-axis, tangent at C makes 0^o with x-axis, tangent at D makes abtuse angle with x-axis, tangent at E makes 0^o with x-axis and tangent at F makes acute angle with x-axis. Consequently the slope of the tangents at A,B,C,D,E,F are respectively positive, positive,, zero, negative, zero and positive.

Hence the portion of the curve $y=f(x)$ which is rising we have $f'(x) > 0$ and for the portion is falling we have $f'(x) < 0$. The points where $f'(x)=0$ i.e. C ties the rising and falling portions and E ties the falling and rising portions.

This conclusion may be obtained analytically. Let c be a point where $f'(c) > 0$. But

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

$$\text{Hence} \quad \lim_{x \to c} \frac{f(x)-(f(c)}{x-c} > 0$$

This shows that $\{f(x) - f(c)\}$ and $(x-c)$ have the same sign for values of x sufficiently close to c. If x lies to the right of c i.e. $x > c$ then $x-c > 0$ and we must have $f(x)-f(c) > 0$ i.e. $f(x) > f(c)$. On the other hand if x lies to the left of c i.e. $x-c < 0$ then we must have $f(x)-f(c) < 0$ and this implies $f(x) < f(c)$. Thus the function has a tendency to increase in the neighbourhood of c. In a like manner we have also $f(x)$ decreasing whenever $f'(x) < 0$. So we have the following theorem.

<u>Theorem</u>: A function is increasing on an interval if $f'(x) > 0$ in it and is decreasing on the interval if $f'(x) < 0$ in it.

Example. Examine the function $f(x) = x^3 - 6x^2 + 9x - 1$ for monotonicity.

$$\text{Here } f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$$
$$= 3(x-3)(x-1).$$

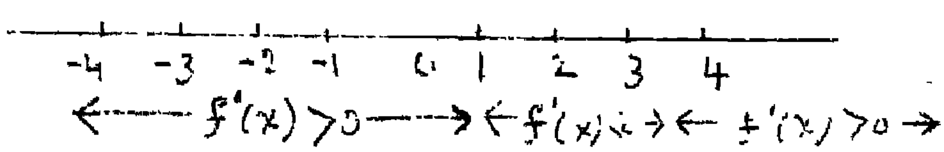
$$f'(x) = 0 \text{ if } x = 3 \text{ and } x = 1.$$

Next we see where the function is increasing and where it is decreasing. Before we do this, we see that

$$x - 1 < 0 \text{ if } x < 1, \quad x - 1 > 0 \text{ if } x > 1$$

$$x - 3 < 0 \text{ if } x < 3, \quad x - 3 > 0 \text{ if } x > 3.$$

It is known that if $f'(x) > 0$, then $f(x)$ is increasing and if $f'(x) < 0$, then $f(x)$ is decreasing. Let us study the situation through the number scale



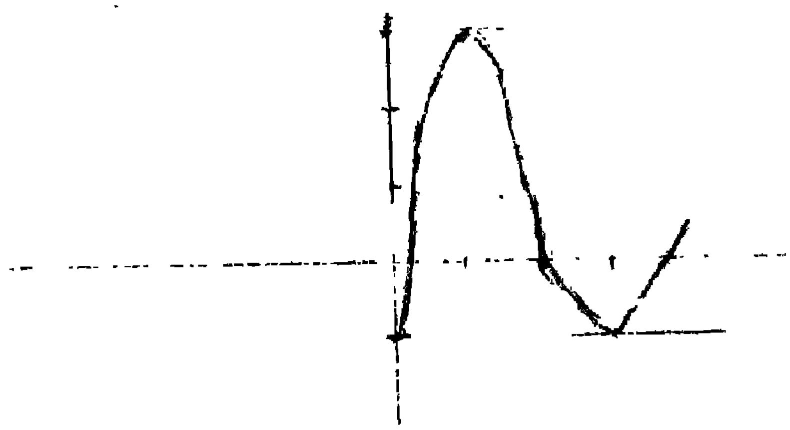
As $f'(x) = 3(x-1)(x-3)$ for $f'(x) > 0$ we should have $(x-1)$ and $(x-3)$ to be of one sign and for $f'(x) < 0$ these two factors should have oppesite signs.

$x > 3 \Rightarrow x > 1$. So for $x > 3$, $f'(x) > 0$. In $1 < x < 3$, $x > 1$ but $x < 3$. So $f'(x) < 0$.

Further $x < 1 \Rightarrow x < 3$. So $f'(x) > 0$.

Hence the function $f(x)$ is increasing for $x > 3$ and $x < 1$ and decreasing in $(1, 3)$. At $x = 1$ and 3 the function has zero slope or in other words the tangents at these points are parallel to x-axis.

Further $f(1) = 3$, $f(3) = -1$. The function $f(x)$ being a polynomial we can roughly visualize the graph of the function and draw the sketch. Further $f(0) = -1$.



EXERCISE

Examine the following functions for positive, negative and zero slopes and sketch its graph.

1 $f(X) = x^2 + 3x - 4$,

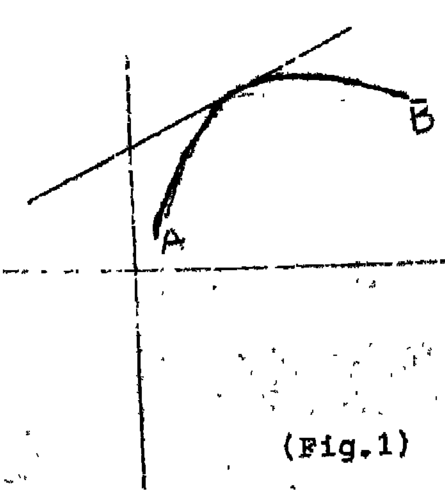
2) $f(x) = (x + 2)(x-4)$

3) $f(x) = x^4 - 9$

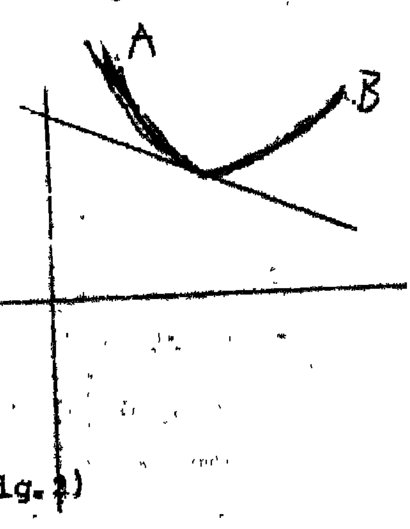
4) $f(x) = 2x^3 - 5x^2 - 4x + 7$.

Concavity of the function:

Let $y = f(x)$ be a function whose graph we can draw. Suppose that $f(x)$ is continuous and twice differentiable in an interval $[a, b]$. When the graph of the function lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$ we say it to be concave in $[a, b]$. However, this is not the rigorous definition. By examining the sign of the second derivative $f''(x)$ we can state the rigorous definition of concavity.



(Fig.1)



(Fig.2)

Let us draw a tangent to the curve at some point (Fig.1). As the point moves from A to B the moving tangent turns clockwise and the slope $f'(x)$ decreases as x increases. In Fig.2 we move from A towards B the tangent turns anticlockwise and the slope increases. Or in other words $f'(x)$ increases. So we have the following:

Definition The graph of the function $y = f(x)$ is said to be concave downwards in an interval if inside it $f'(x)$ decreases as x increases whereas we say the graph $y = f(x)$ to be concave upwards in an interval if $f'(x)$ increases when x increases.

Thus $f''(x) < 0$ in an interval $\Rightarrow y=f(x)$ is concave downwards. $f''(x) > 0$ in an interval $\Rightarrow y = f(x)$ is concave upwards.

Example: Find where the graph $y = x^3 - 6x^2 + 9x - 1=$ is concave downward and where it is concave upward.

Here $y = x^3 - 6x^2 + 9x - 1$

$$y' = 3x^2 - 12x + 9$$

$$y'' = 6x - 12 = 6(x-2)$$

$y'' > 0$ if and only if $x - 2 > 0$ i.e. $x > 2$.
and $y'' < 0$ if and only if $x-2 < 0$ i.e. $x < 2$.
Hence the graph is concave downward for $x < 2$ i.e. to the left of $x = 2$ and concave upward for $x > 2$ i.e. to the right of $x = 2$.

Point of Inflection: A point at which the graph $y = f(x)$ of a continuous function f changes direction of concavity from downward to upward or upward to downward is called a point of inflection. i.e. a point of inflection separates of opposite concavity. Thus if $x = c$ is a point of inflection then in $[c - \delta, c]$ if $y = f(x)$ is concave upward (or downward) then in $[c, c + \delta]$ it is concave downward (or upward) where $\delta > 0$. Evidently we have at

such a point $f''(C) = 0$.

Example:

$$\text{Let } f(x) = \frac{1}{1 + x^2}$$

$$\text{Here } f'(x) = -\frac{2x}{(1 + x^2)^2}$$

$$f'(x) = \frac{2(3x^2 - 1)}{(1 + x^2)^2}$$

$$\text{So } f''(x) = 0 \text{ or } 3x^2 - 1 = 0 \text{ or } x = \pm\sqrt{\frac{1}{3}}.$$

Thus $x = \frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$ are points of inflection on the graph $y = f(x)$.

APPLICABLE MATHEMATICS_I

Dr.K.K.Chakrabarti

INTRODUCTION

1. Most of us know what is meant and implied when we talk about applications of mathematics. We generally mean this to be an exercise where we apply known results and techniques of mathematics to a non-mathematical situation. This is a standard and well-known approach. Let us look at this from a different standpoint. We pose a situation, obviously non-mathematical situation; it may be a part of private life of an individual, may be a workigg life of an individual or drawn from the wider context of social economic, political or cultural life of an individual. Let us catch hold of a situation in which we can have feel of or flavour of mathematics in the sense that there is a case for using mathematical methods; having perceived that, we may put in non-mathematical methods; having perceived that, we may put in non-mathematical language, (but certainly with precision) the statement of the situation. We next pass on to express it, as far as practicable, in terms of mathematical language, using symbols. We may say then that the situation is mathematically formulated. The known techniques and method are then resorted so as to obtain solution of the mathematical problem posed above (this often requires some approximations, assumptions etc.). Next, we draw conclusions from the findings and see if these are in accord with realities of the situation. If not fully, we may have to look afresh at the formulation vis-a-vis solution and if necessary, recast the whole so as to mirror reality better than before. The

../..

totality of the steps is often called mathematization of a situation or mathematical modelling of a situation. It is thus a continuing affair going back and forth from reality so as to obtain better understanding of and insight into the realistic situations. One does not just cease by applying some techniques of mathematics; one goes on to pose the situationwhich assumes variety and complexity from time to time and hence, one has to see on every occasion if mathematics is applicable or not.

In this lecture, we draw upon a situation in our study of environments. We take a situation from what is called population ecology in which an important concern is to study behaviour and dynamics of populations.

2. Statement and Motivation

In order that the sibuation vis-a-vis the problem may seem relevant for introduction in class rooms, one has to provide some motivation for the same.

We may begin with figures in the following table showing production of Houseflies (Muscadomestica) in one year on the assumption that female lays 120 eggs per generation, thathalf of these eggs develop into females, and that there are seven generations per year.

..../....

Generation	If all survive but 1 generation	If all survive 1 year but produce only once	If all survive 1 year and all females produce each generation
1.	120	120	120
2.	7,200	7,320	7,310
3.	432,000	439,320	446,520
4.	25,920,000	26,359,320	27,237,720
5.	1,555,200,000	1,581,559,320	1,661,500,920
6.	93,312,000,000	94,893,559,320	101,351,520,120
7.	5,598,720,000.000	5,693,613,559,320	6,182,442,727,320

Here is an example which shows how two forces operate in the growth and development of population. One of these is the inherent characteristic of each population so as to reproduce at a given rate, opposing which is the inherent capacity for death. Beside death, there are other forces of the physical and biological enviornment in which a population exists. It is the American ecologist Royal Champman, who called these opposing forces in late twenties of the century as 'biotic potential' and 'environmental resistance' respectively.

../..

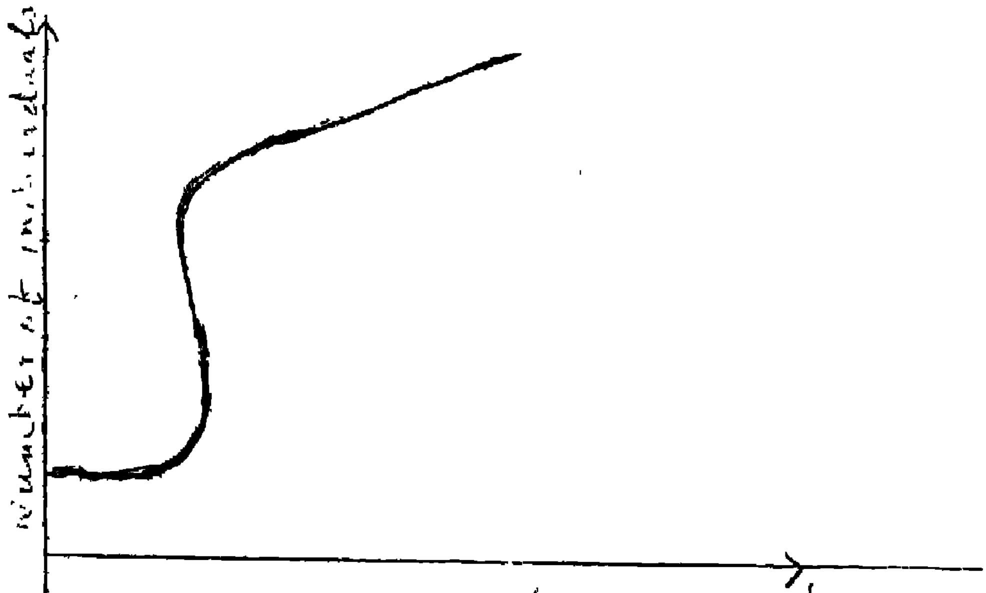
Historically speaking, the study of growth of population was initiated first by Thomas Robert Malthus (1766- 1834), according to whom that the humans could only persist if period of exponential growth were punctuated by plague and famine. It was, in thirties, that Raymond Pearl sought to study theoretical growth of the House Fly so as to fit empirical data on yeast, obtained by a German Investigator T. Carlson,. Pearl converted yeast cells into numbers on the basis of data given below:

Growth of yeast cells in Laboratory Culture

Time (t) (hr.)	Numbers of individual	Increase $\frac{\Delta n}{\Delta t}$
0	9.6	0
2	29.0	19.4
4	71.1	42.1
6	174.6	103.5
8	350.7	176.1
10	513.3	162.6
12	594.4	81.1
14	640.8	46.4
16	655.9	15.1
18.	661.8	5.9

By plotting and joining the points of the first two columns of the above table, we could obtain what is

known as grouth curve of population, which is S-shaped or sigmoidal as shown below.



Grouth Curve of yeast cells

It is curve/table which tells us about the rate of grouth of the population, the period of increase and also of the period of equilibrium which, in fact, is the state where there is no net change in population. The 'carrying capacity' of the environment is often referred to in this connection.

3. A Mathematical model of grouth and decay

We consider the growth of bacteria in a laboratory where the rate of growth is directly proportional to the size of the population. Let us assume the rate of grouth of bacteria to be 20%. Since bacteria divide, the number increases with time. First an initial count is made; let N(1) be the number of bacterial one day after the initial count, then

$$\frac{N(1)}{N(0)} = 1.2$$

If $N(t)$ be the number of bacteria 't' days after the initial count, then $\frac{N(t+1)}{N(t)} = 1.2$

(Since tne increase per day is same)

$$\frac{N(t+1)}{N(t)} = \frac{N(1)}{N(0)},\ \frac{N(t+1)}{N(t-1)} = \frac{N(t+1)}{N(t)} \cdot \frac{N(t)}{N(t-1)} = \frac{N(1)}{N(0)} \cdot$$

$$\frac{N(1)}{N(0)} = \left(\frac{N(1)}{N(0)}\right)^2$$

So $\frac{N(t)}{N(0)} = \left(\frac{N(1)}{N(0)}\right)^t$ for t days.

If we replace 1 day by say, s days then $\frac{N(t+s)}{N(t)} = \frac{N(s)}{N(0)}$ ....(1)

$= \frac{N(1)}{N(0)})^s$. For all positive values of t and s .

The value of this common ratio would be different from 1.2 if s were different from 1, even with same colony of bacteria. Now the relation (1) can be written as

$$N(t+s) = \frac{N(t)\ N(s)}{N(0)} = \left(\frac{N(1)}{N(0)}\right)^s \cdot N(t)$$

Dividing both sides by $N(0)$, we have

$$\frac{N(t+s)}{N(0)} = \frac{N(t)}{N(0)} \cdot \frac{N(s)}{N(0)}$$

Which is symmetrical in s and t.

If we write $\frac{N(t)}{N(0)} = f(t)$, $\frac{N(s)}{N(0)} = f(s)$ and

$$\frac{N(t+s)}{N(0)} = f(t+s)$$

then $f(t+s) = f(t) \cdot f(s)$ ....(2)

This is what is called a functional equation.

Let us now seek the form of f for all real values of s and t. If we try to find the solution at random and try $f(t) = t^2$.

then $f(s) = s^2$ and

$f(t + s) = (t + s)^2 = t^2 + 2st + s^2$

But $t^2 + s^2 + 2\,ts \neq t^2s^2$ for any real values of s and t, even for $s = 1$ and $t = 1$.

If we consider the solution(2) for which $f(t) = c$ where c is a constant, then

$c = c,c = c^2$

This is satisfied only for the values of c to be 0 add 1. But these values are not admissible in this particular bacter.a problem.

Let us take f(t) as a polynomial of degree $n \geqslant 1$, say, p(t). Then $P(t + s) = P(t).\,p\,(s)$.
If $t = s$, then $P(2t) = P^2(t)$ which shows that right hand side is a polynomial of degree 2n and left hand side is a polynomial of degree n, which is impossible.

If we put $s = 0$, then $t + 0 = t$, and then equation (2) becomes.

$$f(t) = f(t).f(0) \quad \ldots\ldots\ldots\ldots(3)$$

Since $f(t) \neq 0$, we can divide the equation (3) by f(t). Hence $f(0) = 1$ which, of course, satisfies the relation

$$f(0) = \frac{N(0)}{N(0)} = 1.$$

If $s = -t$, then

$$f\left[t + (-t)\right] = f(t)\, f(-t)$$

$$\text{or, } f(0) = f(t) \cdot f(-t).$$

Then $f(t)\, f(-t) = 1$ $\quad \therefore \quad f(0) = 1.$

But $f(t) \neq 0$, $f(-t) \neq 0$, hence

$$f(-t) = \frac{1}{f(t)} .$$

Again, if we take

$$t = s = \frac{u}{2} \text{ in (2), then}$$

$$f\left(\frac{u}{2} + \frac{u}{2}\right) = f\left(\frac{u}{2}\right) f\left(\frac{u}{2}\right)$$

$$\text{or, } f(u) = \left[f\left(\frac{u}{2}\right)\right]^2 .$$

Since a square of a number can never be negative,

$$f(u) \geqslant 0.$$

But $f(u) = 0$ is not possible.

Hence $f(u) > 0$.

All values of f are necessarily positive.

If $f(1) = a$, then

$$f(2) = f(1+1) = f(1)\, f(1) = a.a. = a^2$$

$$f(3) = f(2+1) = f(2)\, f(1) = a^2.a = a^3$$

....

By induction, we can show $f(n) = a^n$, when n is any positive integer.

Now, $f(-n) = \dfrac{1}{f(n)} = \dfrac{1}{a^n} = a^{-n}.$

Hence there is possibility of the function f to be of the form

$$f(t) = a^t, \text{ for all real numbers } t.$$

We can prove that this relation is time for any rational value of t.

If we do not consider the irrational values of t we have

$$f(t) = a^t$$

Where t is any rational number. Thus we have the mathematical model for bacterial growth.

Since $$f(t) = \frac{N(t)}{N(0)},$$

we have $$\frac{N(t)}{N(0)} = a^t \quad \therefore \; N(t) = N(0)\, a^t.$$

Hence the number of bacteria at time 't' is the number at time 0, multiplied by the n-th power of 'a', when 'a' is a certain positive number.

But we assumed $$\frac{N(t)}{N(0)} = \left(\frac{N(1)}{N(0)}\right)^t = (1.2)^t$$

Hence $N(t) = N(0)\,(1.2)^t$ can be taken as the mathematical model for the experiment.

The accuracy of this relation can be tested by takang logrithm of both sides and the graph representing this is a straight line.

For,

$$\log N9t) = \log N(0) + t.\log(1.2)$$

which we can write as

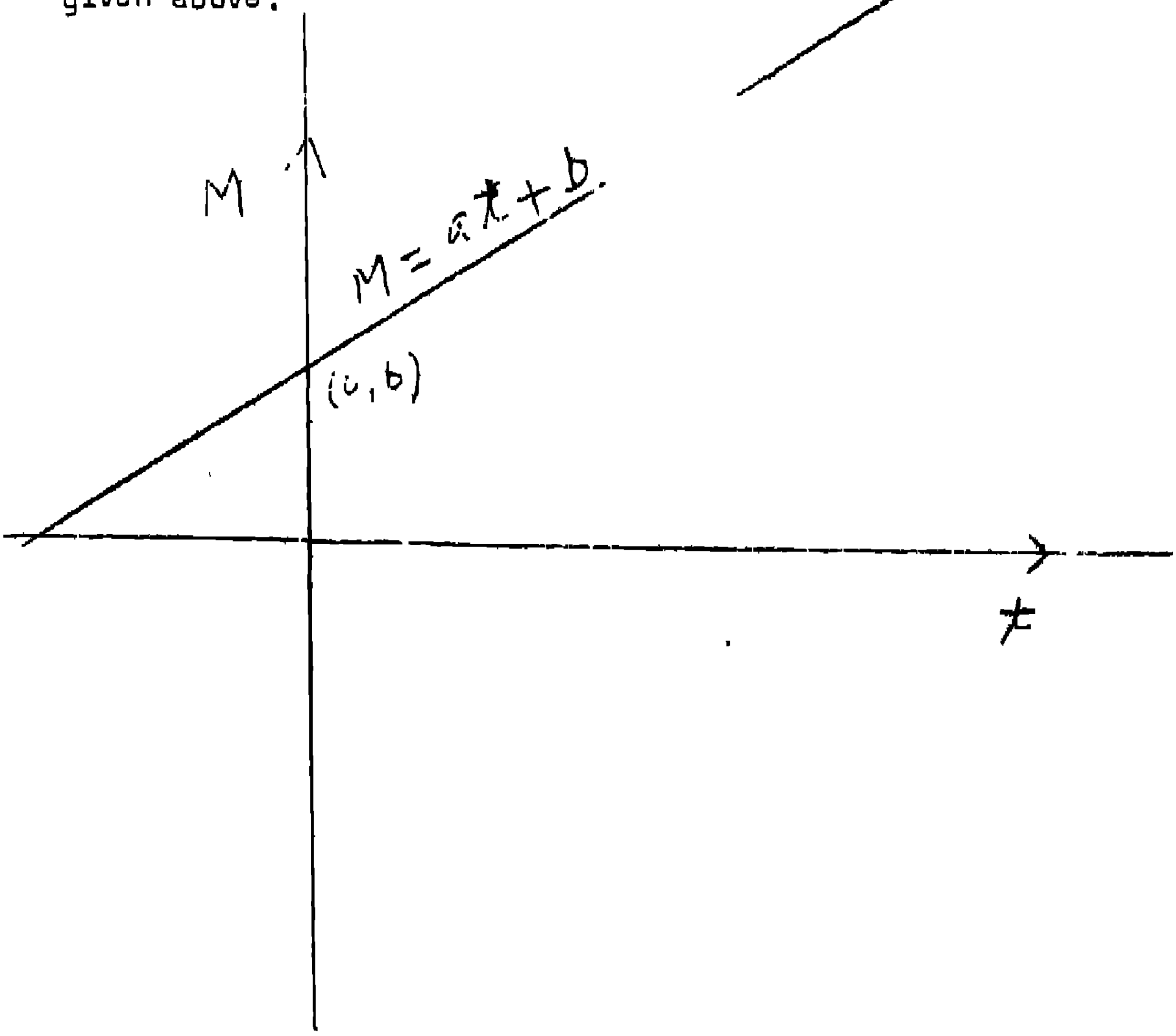
$$M = at + b$$

(with $a = \log(1.2)$

$b = \log N(0)$

$\log N(t) = M(t)$

This is the equation of a straight line with a slope given above.



This is fairly in accord with reality; in fact, if the logarithms of the numbers of bacteria observed at different times 't' are plotted against 't', the points are found to be on a straight line. Hence the model is

$$N(t) = N(0)\, a^t.$$

Pedagogically speaking, this can be taken as a good introduction to the study of indices.

APPLICABLE MATHEMATICS- II

Dr.K.K.Chakravarti

1. INTRODUCTION

We have seen in Lecture I how populations undergo changes in size and we made use of a functional equantion so as to arrive at a form representing the law of growth of a population. We now look at it from a different angle and use another technique of mathematics so as to shed better light on the understanding of the phenomenon involved. We would use, in fact, a differential equation as a means to get at the solution of the mathematical model posed for growth of a single species population.

2. PROBLEM AND ITS SOLUTION: FIRST MODEL

It is a reality of many populations particularly human populations that there is a overlap between generations, so that one can justifiably take the population changes to be continuous in nature. In view of this, we can formulate the problem as follows. Let, as before, $N(t)$ be the population size at time t and let $N(t + \Delta t)$ be the same at time $(t + \Delta t)$.

Then we can take the increase in the population size in the small time interval $(t, t + \Delta t)$ to be proportional to the population size $N(t)$ at time t and the length of the interval i.e., Δt. We have then

(1) $N(t + \Delta t) - N(t) = (b-d)\, N(t)\, \Delta t$

where b and d, as before, are the constant birth and death rates of the poulation. Dividing both sides of (1) by Δt, and proceeding to the limit as $t \to 0$, we get

(2) $$\frac{dN}{dt} = rN$$

where r(=b -d) is the biotic potential or intrinsic growth rate. A variant of this equation (2) may be

$$\frac{dN}{dt} = bN$$

where b is the constant birth rate; another similar form may be

$$\frac{dN}{dt} = -dN$$

where d is the constant death rate.

One can make a guess of the solution of equation (2) as we are already aware of a similar form in Lecture 1. Indeed, one may write immediately.

$$N = N_0 e^{rt}$$

and verify the same with the equation (2). Let us, however, obtain a formal solution of equation (2) as follows. We write equation (2) as

$$\frac{dN}{N} = r\, dt.$$

by separating the variables. Integrating this, we have

$$\log N = rt + \text{constant}.$$

Let $N = N_0$ at $t = 0$, so that constant $= \log N_0$

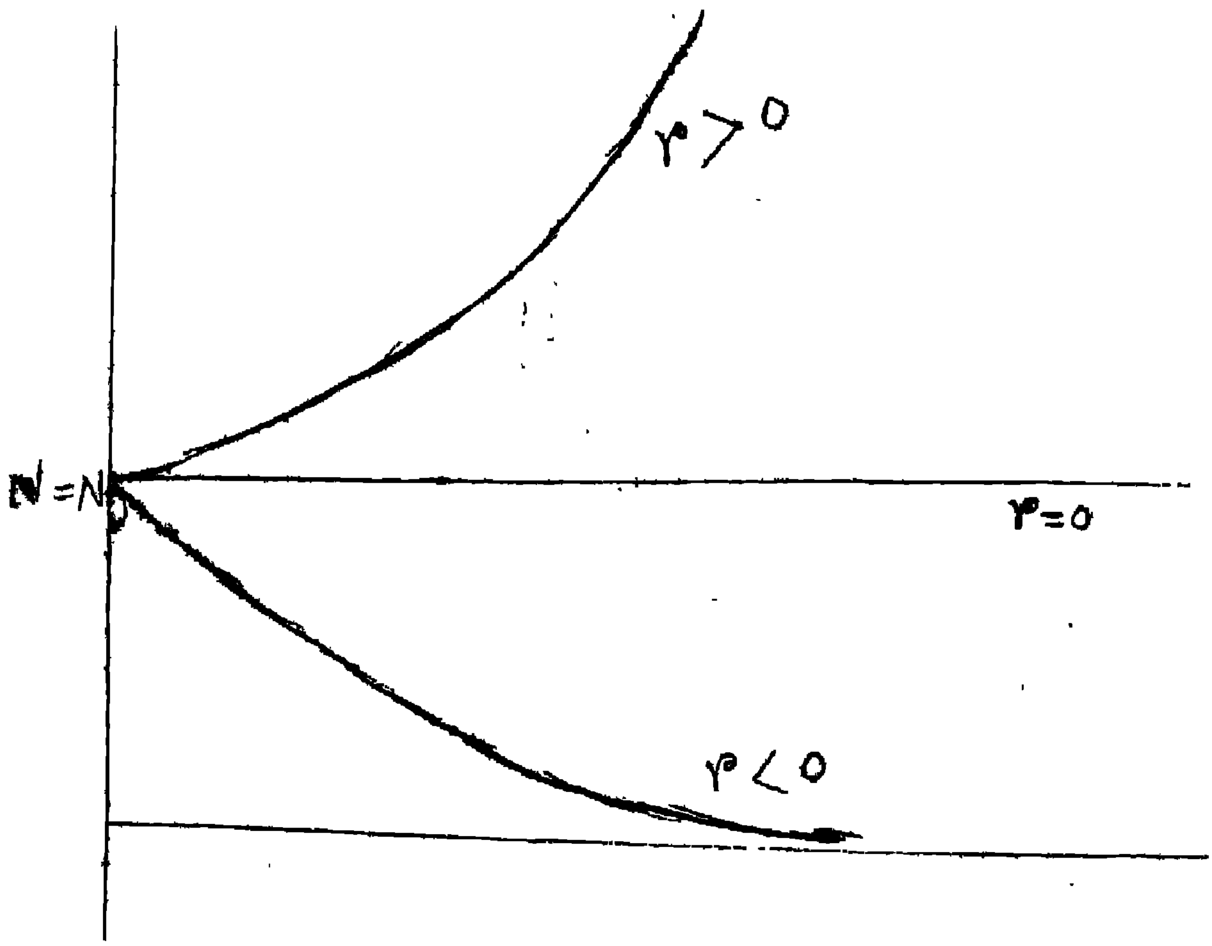
$$\therefore \log N = rt + \log N_0$$

$$\text{or, } \log N - \log N_0 = rt.$$

$$\text{or, } \log \frac{N}{N_0} = rt$$

(3) $$\therefore N = N_0 e^{rt}$$

Let us now analysis this equation by drawing the graph of it as shown below.



Obviously for r = 0 i.e. birth rate equal to death rate, the poulation has the uniform value N_0 throughout. For r < 0 , i.e. birth rate less then death rate, the population size suffers an exponential decay with time.

For r > 0 i.e. birth rate greater than the death rate, the population size has a tendency to increase beyond all bounds as time increases. For both r > 0 and r < 0, we call the behaviour to be one of exponential in nature. It is the exponential character, particularly the one which leads to unbounded growth, that calls for re-examination of the model in the light of reality. The model does, of course, predict.

population sizes for small time scales and is totally inadequate for longer time scales. Further, one has to take into account, a fact of nature, that every species of organism has a finite amount of space, food and also limited supply of resource; in fact, one has to set a upper limit on the number of individuals that can exist on the available resources. This upper limit of the number of populations, in ecological parlance, has been called 'carrying capacity' of the environment; evidently, as the number of population approaches the 'carrying capacity', the rate of growth decreases. Any realistic model of population behaviour has to reckon this aspect of environment. Hence we need to modify the model.

3. PROBLEM AND ITS SOLUTION: SECOND MODEL

Let K be the carrying capacity of the environment. If, as before, N be the population size at any time t, then (K-N) is the difference between the upper limit and the population size at time t. As population grows and approaches the upper limit, more resistance is encountered on account of limitations mentioned above. One way of representing this environmental resistance is to take this as $\frac{(K-N)}{K}$, which, by assumption, acts against (rN). Therefore, the rate of change of population size can be taken

$$(4) \qquad \frac{dN}{dt} = rN \left(\frac{K-N}{K}\right) = rN \left(1 - \frac{N}{K}\right)$$

This equation is often called logistic equation; the model, called logistic model or verhulst equation

named after Verhulst. This was obtained independently by Pearl and Reed in early twenties of this century in their attempt to obtain a general formula for population growth by fitting empirically a curve/ data. Hence equation (4) is called Pearl-Verhulst equation.

Let us seek the solution of (4) as follows. We write (4) as:

$$\frac{dN}{dt} = N(r - aN)$$

where

$$(5) \qquad a = \frac{r}{K}$$

Using partial fractions separating the variables and integrating we have then

$$\frac{a}{r} \int \left[\frac{1}{N} + \frac{1}{(r/a - N)} \right] dn = a \int dt$$

or, $\log \left(\frac{N}{(r/a - N)\, C} \right) = rt$ by cancelling a

Where C is a constant obtainable readily when N_0, the population size, is given at t = 0, We can write the last equation as

$$N = \left(\frac{r}{a} - N \right) C \; e^{rt}$$

$$\text{or, } \; N(1 + C\, e^{rt}) = \frac{r}{a} C\, e^{rt}$$

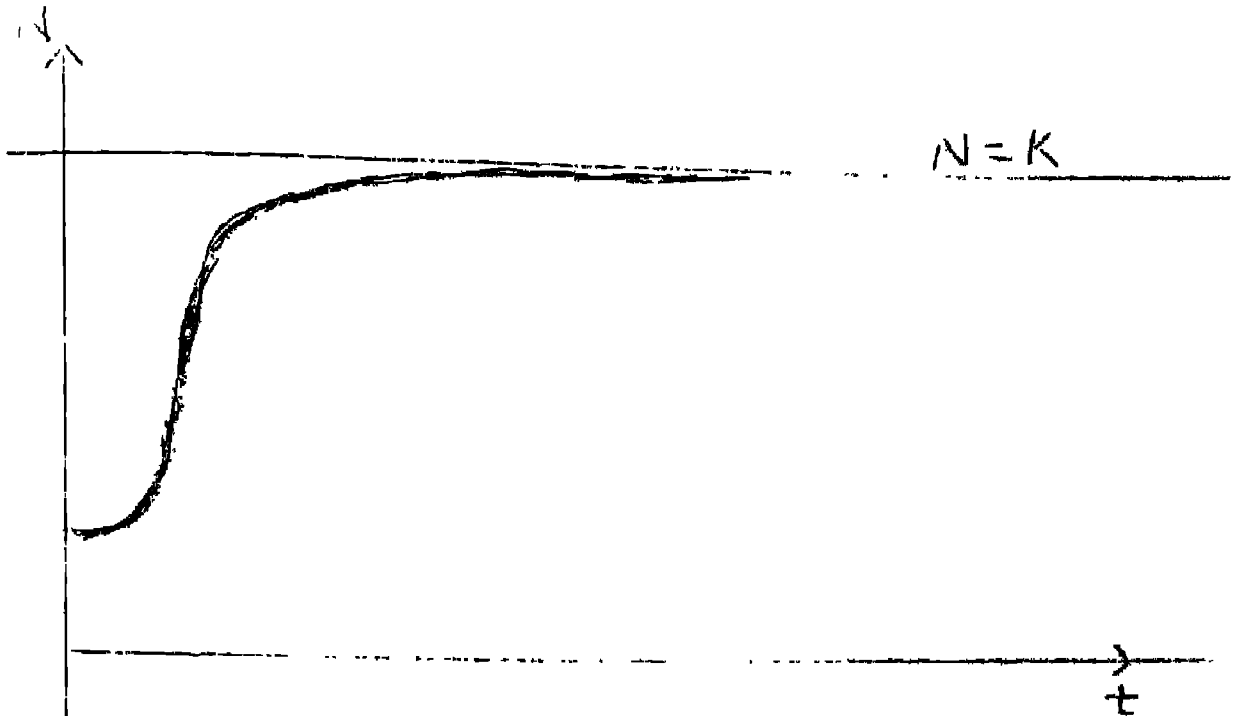
$$\text{or, } \; N(1 + C\, e^{rt}) = \frac{r}{a} C \; e^{rt}$$

$$\text{or, } \; N = \frac{r/a\; C\, e^{rt}}{1 + C\, e^{rt}}$$

$$(6) \qquad \therefore \; N(t) = \frac{r/a}{\frac{1}{C}\, e^{-rt} + 1} = \frac{K}{\frac{e^{-rt}}{C} + 1} \quad \text{by (5)}$$

16

as $t \to \infty$, $N(t) \to K$ which is the carrying capacity of the environment. The curve, known as logistic curve, representing equation (6) is as follows:



This, as we have seen earlier, looks like an elongated S. The model predicts a saturation value (K) to which population size will attain for larger time intervals. The model does reflect reality for some kinds of species, e.g. yeast cells which we have met earlier. But it does not reflect reality in its totality. The limitations and allied aspects are taken up in the next section.

4 SOME REMARKS

The basic disability of the model continues, because of our consideration of a single species population a single species in isolation from others is surely remote from reality but its theoretical consideration

does help in getting some insight into the problem. We shall move over to more than one species in the next lecture. Nevertheless, one can look deeply into the equation (2), in which b and d are taken to be uniform. In order that the model becomes more realistic, we need to take both b and d as functions of population size, i.e. N andhence equation (2) is to be replaced by

$$(7) \quad \frac{dN}{dt} = N\,f(N)$$

One can, mathematically, now think of variants of (7) by giving to f(N) some suitable expressions. For example if we take

$$f(N) = (a-bN)$$

where $a > 0$, $b > 0$, (Note: $\frac{d}{dN} f(N) = -b < 0$

we have

$$(8) \quad \frac{dN}{dt} = N(a-bN)$$

which is another version of Logistic equation (a,b are called logistic parameters).

By taking

$$f(N) = -\lambda \log\left(\frac{N}{\theta}\right)$$

where λ and θ are constant, we have another model of a single species population, known as Gompertzian model. In fact we have then

$$\frac{dN}{dt} = -\lambda N \log\left(\frac{N}{\theta}\right)$$

or $$\int \frac{d(\log N)}{\log N - \log\theta} = -\lambda \int dt$$

or, $\log(\log N - \log\theta) = -\lambda t + \text{constant}.$

Let $N = N_0$ at $t = 0$, then

$$\log(\log N_0 - \log\theta) = \text{constant}$$

which lead to

$$\log\left[\frac{\log N - \log\theta}{\log N_0 - \log\theta}\right] = -\lambda t$$

or

$$\left[\frac{\log N - \log\theta}{\log N_0 - \log\theta}\right] = \exp(-\lambda t)$$

or, $$= 1 - \frac{\log N - \log\theta}{\log N_0 - \log\theta} = (1 - \exp(-\lambda t))$$

or, $$\frac{\log N_0 - \log N}{\log N_0 - \log\theta} = .\ (1 - \exp(-\lambda t))$$

or, $$\log\frac{N}{N_0} = \log\frac{\theta}{N_0}(1 - \exp(-\lambda t))$$

$$\therefore \frac{N}{N_0} = \exp\left(\log\frac{\theta}{N_0}(1 - \exp(-\lambda t))\right)$$

We end up this lecture with two remarks. The equation (2) may be further complicated but made close to reality if one takes into account time-lag τ which leads to a time-delayed differential equation of the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N(t-\tau)}{K}\right)$$

which is beyond present scope of lectures.

The other remark is that it is not populations, human or bird or animals or plants, only whose behaviour is modelled in the above ways; of late, growth of tumour cells in the study of cancer treatment obeys similar models.

Pedagogically speaking, the consideration of these models provide appropriate motivation and introduction to rates of changes in terms of derivatives, differential equations and their solutions.

APPLICABLE MATHEMATICS - III

Dr.K.K.Chakrabarti

1. INTRODUCTION

We continue in this lecture, the study on dynamics of populations of more than one species; in fact, for reasons of simplicity and brevity, we confine ourselves to two species populations. This is referred to as two species models. As already remarked, no organism can live in complete isolation from the other(s) and often interact with each other for their survival. They complete for such limited resources as food, space, etc. There may be cases where one may kill the other for food, for protection; etc. as a matter of fact, one species (called predator/parasite) may have the other species (Called prey/host) as the food. Let us consider this predator - prey (P-P)/host-parasite (H-P) interactions. The animal kingdom is replete with P-P/H-P interactions. A simple example is given by populations of foxes (predators) being fed by rabbits (preys).

2. PREY-PREDATOR (P-P)/HOST-PARASITE (H-P) MODELS

The model is developed on the basis of following assumptions:

(i) in isolation, the rate of change of size of one population is proportional to the size of the population; for example, if there is no fox, the rabbit population will

grow exponentially; if there be no rabbit, the fox population will undergo a pure death (exponential) process.

(ii) the only food available to the predators (foxes) are the preys (rabbits).

(iii) the number of kills of prey (rabbits) by predator (foxes) is proportional to the frequency of encounters between them, which, in turn, is taken to be proportional to the product of population of prey and predator. Obviously, the number of kills will be more, if both populations are large and if the number of prey and predator be less, the number of kills is few.

Let x denote the prey population at time t and y the predator population. The model is developed by Vito-Volterra and is given by pair of first order differential equations, in which x and y are both functions of time t. These are

$$(1) \qquad \frac{dx}{dt} = ax - bxy$$

$$(2) \qquad \frac{dy}{dt} = mxy - ny$$

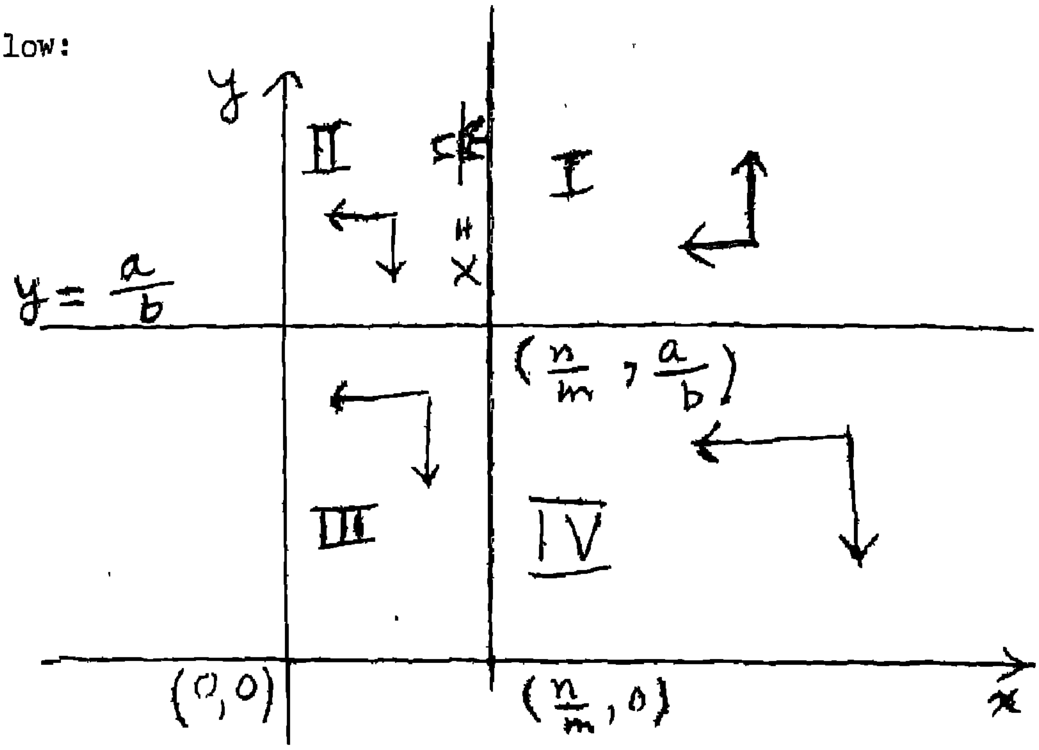
where a, b, m, n are positive constants. We find from (1) and (2),

$$\frac{dx}{dt} = 0 \text{ when } x = 0, \ \& \ y = \frac{a}{b}$$

$$\frac{dy}{dt} = 0 \text{ when } y = 0 \ \& \ x = \frac{n}{m}$$

We may recall that $(0,0)$, $\left(\frac{a}{b}, \frac{n}{m}\right)$ are critical or equilibrium points of the system. Let us now analyse graphically the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ with a view to indicating how rate of population sizes behave with

respect to time. Obviously, the lines $y = \frac{a}{b}$ and $x = \frac{n}{m}$ divide the first quadrant into four regions as in the figure below:



From equation (1), we can say that $\frac{dx}{dt}$ is positive if $y < \frac{a}{b}$ and negative if $y > \frac{a}{b}$, while from (2) $\frac{dy}{dt}$ is positive when $x > \frac{n}{m}$ and negative when $x < \frac{n}{m}$.

Let us now see what take place in the regions I, II, III, IV of the first quadrant. We can possibly describe this as follows :

	Region	$\frac{dx}{dt}$	$\frac{dy}{dt}$
I	$(y > \frac{a}{b}, x > \frac{n}{m})$	'–' ve	'+' ve
II	$(y > \frac{a}{b}, x < \frac{n}{m})$	'–' ve	'–' ve
III	$(y < \frac{a}{b}, x < \frac{n}{m})$	'+' ve	'–' ve
IV	$(y < \frac{a}{b}, x > \frac{n}{m})$	'+' ve	'+' ve

The arrows in each of the four regions indicate the drift of the path of the orbit of the system.

At a certain time, both the predator and prey population experience a growth in numbers, as is shown in region IV. When the predator (fox) exceeds the critical level $\frac{a}{b}$, the predators eat the prey whose numbers begin to decline as is shown in region I. When the number of prey population decreases and its size falls below $\frac{n}{m}$, as in region II, then there is not a sufficient supply of prey to sustain a large predator population. Both the species then lose. Gradually, as in region III, the reproduction rate of prey overtakes its disappearance rate and the prey population recovers. The surviving predators find their food again, thereby survive and the whole cycle recurs, unless at some previous stage, one of the populations have vanished altogether. This kind of analysis gives an idea about the oscillatory behavious of the system. Historically speaking, such variations in numbers and indeed, a cyclic pattern was observed by the zoologist D. Anconna in course of researches of fish populations caught in the Adriatic during World Was I and he took it to the mathematician Vito Volterra for a mathematical explanation. Volterra obtained a mathematical model of these relationships and later on, it was done by A. Lotka. So the models on two species interaction of this sort, go by the name. Lotka-Volterra (L-V) models.

Let us rewrite the equations (1) & (2) in a different form, writing H for x and P for y, (H - the host, P - parasite). Hence the H - P equations are

(4) $$\frac{dH}{dt} = (a - bP)H$$

(5) $$\frac{dP}{dt} = (-n + mH)P.$$

Before we solve these equations, let us have an idea of the path these equations may represent. To do this, we divide one by the other and have

$$\frac{dH}{dP} = \frac{(a - bP)H}{(-n + mH)P}$$

or, $$\frac{-n + mH}{H} \quad dH = \frac{a - bP}{P} \quad dP$$

or, $$-\frac{n}{H}\,dH + m dH = \frac{a}{P}\,dP - bdP$$

Integrating

$$-n \log H + mH = a \log P - bp + \text{constant}.$$

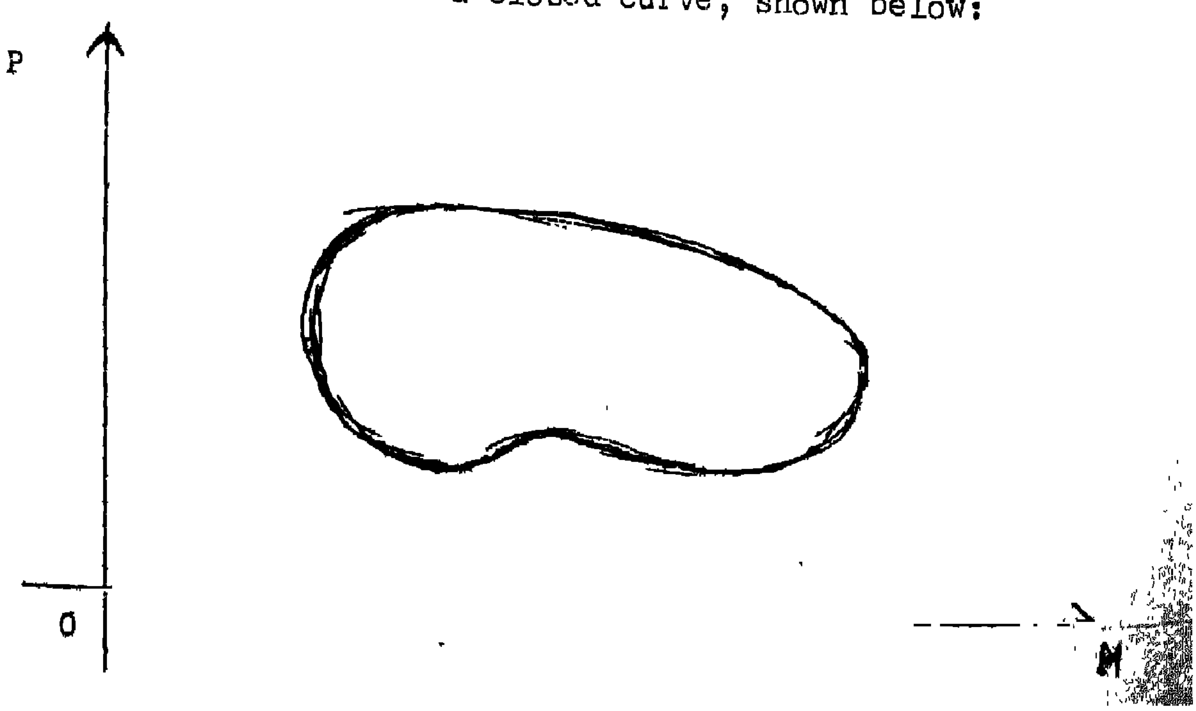
which we can write as

$$\log H^{-n} + \log e^{mH} = \log P^{a} + \log e^{-bP} + \text{constant}$$

which has the form

(6) $$(P^{a}e^{-bP})\,(H^{n}e^{-mH}) = C$$

which can be seen to be a closed curve, shown below:



Let us now try to get a better idea of this path in familiar terms. To do this, we note from (4) and (5) that the nonzero critical point (H_0, P_0) of the system is given by

$$H_0 = \frac{n}{m} \quad , \quad P_0 = \frac{a}{b}$$

The equations (4) and (5) are not easily solvable but we can solve it by using the substitutions

$$(7) \qquad H = H_0 + h$$

$$(8) \qquad P = P_0 + P$$

where h and p are quantities so small that their squares and higher powers may be neglected.

Equations (7) and (8) have special meanings in the sense that we are, indeed, seeking for values of H and P in the neighbourhood of the critical point $(\frac{n}{m}, \frac{a}{b})$ which we ought to do as we have observed graphically, earlier some fluctuating behaviour. Putting (7) and (8) in (4) and (5) we get, on simplification :

$$\frac{dh}{dt} = a(H_0 + h) - b(P_0 + p)(H_0 + h)$$

$$= aH_0 + ah - bP_0H_0 - bP_0h - bP_0p - b\cancel{p}h$$

$$= \cancel{a\frac{n}{m}} + \cancel{ah} - \cancel{b\frac{na}{bm}} - \cancel{b\cdot\frac{a}{b}h} - b\cdot\frac{n}{m}\cdot p = -\frac{bn}{m}p$$

(neglecting ph)

$$\frac{dp}{dt} = -n(P_0 + p) + m(H_0 + h)(P_0 + p)$$

$$= -nP_0 - np + mH_0P_0 + mH_0p + mP_0h + \cancel{mhp}$$

$$= -\cancel{n\frac{a}{b}} - \cancel{np} + \cancel{m\frac{na}{mb}} + \cancel{m\cdot\frac{n}{m}\cdot p} + m\cdot\frac{a}{b}\cdot h = \frac{ma}{b}h$$

(neglecting ph)

$$(\because \; F_0 = \frac{n}{m} \; , \; P_0 = \frac{a}{b} \;)$$

Which we can write as

$$(9) \quad \frac{dh}{dt} = - \frac{bn}{m} \; p$$

$$(10) \quad \frac{dp}{dt} = \frac{ma}{b} \; h$$

From (9) and (10), we get

$$\frac{dh}{dp} = - \frac{bn}{m} \times \frac{b}{ma} \; \frac{p}{h}$$

$$\text{or} \quad \frac{hdh}{b^2n} + \frac{pdp}{m^2a} = 0$$

Integrating

$$(11) \quad \frac{h^2}{(\sqrt{n}b)^2} + \frac{p^2}{(\sqrt{a}m)^2} = \text{constant}$$

which are ellipses in (H-P) plane so that one may conclude that about the critical point paths described are ellipses.

From (9) and (10)

$$\frac{d^2h}{dt^2} = - \frac{bn}{m} \; \frac{dp}{dt} = - \frac{\cancel{b}n}{\cancel{m}} \cdot \frac{\cancel{m}a}{\cancel{b}} \; h$$

$$(12) \quad \frac{d^2h}{dt^2} + nah = 0$$

and

$$\frac{d^2p}{dt^2} = \frac{ma}{b} \; \frac{dh}{dt} = \frac{\cancel{m}a}{\cancel{b}} \cdot \frac{\cancel{b}n}{\cancel{m}} \; p$$

$$(13) \quad \text{or,} \quad \frac{d^2p}{dt^2} + nap = 0$$

Equations (12) and (13) show that both h and n satisfy the same equation, namely

$$(14) \quad \frac{d^2x}{dt^2} + \mu^2 x = 0$$

Where μ^2 = an

Setting $\frac{dx}{dt} = v$, $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dx}(v) \cdot \frac{dx}{dt} = v\frac{dv}{dx}$

Equation (14) has the form

$$(15) \quad v\frac{dv}{dx} = -\mu^2 x$$

Integrating,

$$\frac{v^2}{2} + \frac{\mu^2 x^2}{2} = \text{constant}$$

which mean

$$(16) \quad \left.\begin{array}{l} \frac{1}{2}\left(\frac{dp}{dt}\right)^2 + \frac{\mu^2 p^2}{2} \\ \frac{1}{2}\left(\frac{dh}{dt}\right)^2 + \frac{\mu^2 h^2}{2} \end{array}\right\} = \text{constant.}$$

This is somewhat analogous to the energy equation in a mechanical system. We may conclude from (16), that there exists a constancy of the dynamics of host-parasite/prey-preiator interactions. Equation (15) has the solution given by

$$(17) \quad x = a \cos \mu t$$

Where a is the amplitude. h and p can thus be computed and H and P become known. These show analytically that in the neighbourhood of the critical point there exists

an oscillatory motion, having the period $\frac{2\pi}{\mu} = \frac{2\pi}{\sqrt{an}}$, as predicted qualitatively by the graphical analysis undertaken above.

Let us investigate if the original equations exhibit any constancy of the system. To do this, we write V-P equations as.

(18) $$\frac{dN_1}{dt} = a_{11}N_1 - a_{12}N_1N_2$$

(19) $$\frac{dN_2}{dt} = a_{21}N_2 + a_{22}N_1N_2$$

$$\frac{dN_1}{dt} = 0 = \frac{dN_2}{dt}$$ give respective equilibrium Values of N_1 and N_2 as

(20) $$q_1 = \frac{a_{21}}{a_{22}}, \quad q_2 = \frac{a_{11}}{a_{12}}$$

Let (21) $$N_1 = q_1 \exp(v_1)$$

(22) $$N_2 = q_2 \exp(v_2)$$

Then (18) and (19) become

$$\frac{dv_1}{dt} = a_{11} - a_{12}q_2 \exp(v_2) = a_{11} - a_{11} \exp(v_2)$$

$$\frac{dv_2}{dt} = - a_{21} + a_{22}q_1 \exp(v_1) = -a_{21} + a_{21} \exp(v_1)$$

Which we can write

$$\frac{1}{a_{11}} \frac{dv_1}{dt} = 1 - \exp(v_2)$$

$$\frac{1}{a_{21}} \frac{dv_2}{dt} = - 1 + \exp(v_1) = - \left[1 - \exp(V_1)\right]$$

from which we get

$$\frac{1}{a_{11}}\left[1 - \exp(v_1)\right] \frac{dv_1}{dt} + \frac{1}{a_{21}}\left[1 - \exp(v_2)\right] \frac{dv_2}{dt} = 0$$

or, $$\frac{d}{dt}\left[\frac{1}{a_{11}}\left\{v_1 - \exp(v_1)\right\} + \frac{1}{a_{21}}\left\{v_2 - \exp(v_2)\right\}\right] = 0$$

Integrating,

$$a_{21}\left\{v_1 - \exp(v_1)\right\} + a_{11}\left\{v_2 - \exp(v_2)\right\} = \text{constant}$$

or, $$a_{21}\left\{\log \frac{N_1}{q_1} - \frac{N_1}{q_1}\right\} + a_{11}\left\{\log \frac{N_2}{q_2} - \frac{N_2}{q_2}\right\} = \text{constant}$$

or $$\log \left(\frac{N_1}{q_1}\right)^{a_{21}} + \log\left(\frac{-N_1}{e^{q_1}}\right)^{a_{21}} + \log\left(\frac{N_2}{q_2}\right)^{a_{11}} + \log\left\{\frac{-N_2}{e^{q_2}}\right\}^{a_{11}} = \text{constant}$$

which give

$$\left(\frac{N_1}{q_1}\right)^{a_{21}} \left(\frac{N_2}{q_2}\right)^{a_{11}} e^{-\left(\frac{N_1}{q_1}\right)^{a_{21}}} e^{\left(\frac{N_2}{q_2}\right)^{a_{11}}} = \text{consta}$$

which is an integral of the system. This is often called in invariant of the system whatever be the changes in the population sizes.

If two species be competing for common resources, we can proceed similarly and show one species is to survive, one cannot do so without excluding the other. This is often referred to as Ganse's competitive exclusive principle.

Pedagogically speaking, this lecture provides good motivation and introduction to graphical analysis of equation paths represented by them and their integrals (constant).